ELEMENTARY TEXTBOOK

ON THE

CALCULUS

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W. P. 1

PREFACE

The present volume is the outgrowth of the requirements for students in engineering and science in Cornell University, for whom a somewhat brief but adequate introduction to the Calculus is prescribed.

The guiding principle in the selection and presentation of the topics in the following pages has been the ever increasing pressure on the present-day curriculum, especially in applied science, to limit the study of mathematics to a minimum of time and to the topics that are deemed of most immediate use to the professional course for which it is preparatory.

To what extent it is wise and justifiable to yield to this pressure it is not our purpose to discuss. But the constantly accumulating details in every pure and applied science makes this attitude a very natural one towards mathematics, as well as towards several other subjects which are subsidiary to the main object of the given course.

This desire to curtail mathematical training is strikingly evidenced by the numerous recent books treating of Calculus for engineers, for chemists, or for various other professional students. Such books have no doubt served a useful purpose in various ways. But we are of the opinion that, in spite of the unquestioned advantages of learning a new method by means of its application to a specific field, a student would ordinarily acquire too vague and inaccurate a command of the fundamental ideas of the Calculus by this one-sided presentation. While a suitable illustration may clear up the difficulties of an abstract theory, too constant a dwelling among applications alone, especially from one point of view, is quite as likely to prevent the learner from grasping the real significance of a vital principle.

In recognition of the demand just referred to, we have made special effort to present the Calculus in as simple and direct a form as possible consistent with accuracy and thoroughness. Among the different features of our treatment, we may single out the following for notice.

The derivative is presented rigorously as a limit. This does not seem to be a difficult idea for the student to grasp, especially when introduced by its geometrical interpretation as the slope of the line tangent to the graph of the given function. For the student has already become familiar with this notion in Analytic Geometry, and will easily see that the analytic method is virtually equivalent to a particular case of the process of differentiation employed in the Calculus.

In order to stimulate the student's interest, easy applications of the Differential Calculus to maxima and minima, tangents and normals, inflexions, asymptotes, and curve tracing have been introduced as soon as the formal processes of differentiation have been developed. These are followed by a discussion of functions of two or more independent variables, before the more difficult subject of infinite series is introduced.

In the chapter on expansion, no previous knowledge of series is assumed, but conditions for convergence are discussed, and the criteria for determining the interval of convergence of those series that are usually met with in practice are derived.

A chapter on the evaluation of indeterminate forms and three chapters on geometric applications furnish ample illustration of the uses of infinite series in a wide range of problems.

By reason of its significance in applications, it does not seem advisable to omit the important principle of rates. Arising out of the familiar notion of velocity, it affords an early glimpse into applications of the Calculus to Mechanics and Physics. We do not propose to make the Calculus a treatise on Mechanics, as seems to be the tendency with some writers; but a final chapter on applications to such topics of Mechanics as are easy to comprehend at this stage is thought advisable and sufficient. Especially in treating of center of gravity, the formulas have been derived in detail, first for n particles, and then, by a limiting process, for a continuous mass. This was considered the more desirable, as textbooks in applied mathematics frequently lack in rigor in discussing the transition from discrete particles to a continuous mass. Besides, the derivation of these formulas affords a very good application of the idea of the definite integral as the limit of a sum. This idea has been freely and consistently used in the derivation of all applied formulas in the Integral Calculus. However, as the formula for the length of arc in polar coördinates is especially difficult of derivation by this method, we have deduced it from the corresponding formula for rectangular coördinates by a transformation of the variable of integration.

In order to make the number of new ideas as few as possible, the notions of infinitesimals and orders of infinitesimals have been postponed to the last article on Duhamel's principle. This principle seems to flow naturally and easily from the need of completing the proof of the formulas for center of gravity. The teacher may omit this article, but its presence should at least serve the important end of calling the attention of the student to the fact that there is something yet to be done in order to make the derivations complete.

Some teachers will undoubtedly prefer to do a minimum amount of work in formal integration and use integral tables in the chapters on the applications. For such the first chapter of the Integral Calculus might suffice for drill in pure integration. The problems in this chapter are numerous, and, for the most part, quite easy, and should furnish the student a ready insight into the essential principles of integration.

The characteristic features of the books on the Calculus previously published in this series have been retained. The extensive use of these books by others, and a searching yearly test in our own classroom experience convince us that any far-reaching change could not be undertaken without endangering the merits of the book. The changes that have been made are either in the nature of a slight rearrangement, or of the addition of new illustrative material, particularly in the applications.

We wish to acknowledge our indebtedness to our colleagues, who have added many helpful suggestions; to Professor I. P. Church, of the College of Civil Engineering, for a number of very useful problems in applications of integration (See Exs. 14–18, pp. 318–320, and Exs. 6–7, pp. 323–324), and particularly to Professor James McMahon, who has carefully read all the manuscript, assisted throughout in the proof reading, and made many improvements in the text.

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DIFFERENTIAL CALCULUS

CHAPTER I

FUNDAMENTAL PRINCIPLES

1. Elementary definitions. A *constant* number is one that retains the same value throughout an investigation in which it occurs. A *variable* number is one that changes from one value to another during an investigation. If the variation of a number can be assigned at will, the variable is called *independent;* if the value of one number is determined when that of another is known, the former is called a *dependent* variable. The dependent variable is called also a *function* of the independent variable.

E.g., $3x^2$, $4\sqrt{x-1}$, $\cos x$, are all functions of x.

Functions of one variable x will be denoted by the symbols $f(x)$, $\phi(x)$, $\cdots$, which are read as "f of x," "ϕ of x," etc.; similarly, functions of two variables, x, y, will be denoted by such expressions as

$$f(x, y),\ F(x, y),\ \cdots.$$

When a variable approaches a constant in such a way that the difference between the variable and the constant may become and remain smaller than any fixed number, previously assigned, the constant is called the *limit* of the variable.

2. Illustration: Slope of a tangent to a curve. To obtain the slope of the tangent to a curve at a point P upon it, first take the slope of the line joining $P \equiv (x_1, y_1)$ to another point (x_2, y_2) upon the curve, then determine the limiting value of the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

as the second point approaches to coincidence with the first, always remaining on the curve.

Ex. 1. Determine the slope of the tangent to the curve

$$y = x^2,$$

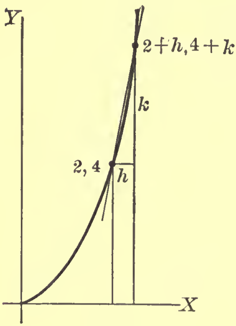

FIG. 1

at the point (2, 4) upon it.

Here, $x_1 = 2$, $y_1 = 4$. Let $x_2 = 2 + h$, $y_2 = 4 + k$, where h, k are so related that the point (x_2, y_2) lies on the curve.

Thus $$4 + k = (2 + h)^2,$$

or $$k = 4h + h^2. \qquad (1)$$

The slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ becomes

$$\frac{4 + k - 4}{2 + h - 2} = \frac{k}{h},$$

which from (1) may be written in the form

$$\frac{k}{h} = 4 + h. \qquad (2)$$

The ratio $k : h$ measures the slope of the line joining (x_1, y_1) to (x_2, y_2). When the second point approaches the first as a limiting position, the first member of equation (2) assumes the indeterminate form $\frac{0}{0}$, but the second member approaches the definite limit 4. When the two points approach coincidence, a definite slope 4 is obtained, which is that of the tangent to the curve $y = x^2$ at the point (2, 4).

It may happen that h, k appear in both members of the equation which defines the slope, as in the next example.

Ex. **2.** If $x^2 + y^2 = a^2$, find the slope of the tangent at the point (x_1, y_1).

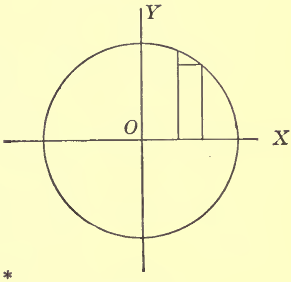

FIG. 2

Since

$x_1^2 + y_1^2 = a^2$, $(x_1 + h)^2 + (y_1 + k)^2 = a^2$,

hence $2hx_1 + h^2 + 2ky_1 + k^2 = 0$,

from which $\dfrac{k}{h} = -\dfrac{2x_1 + h}{2y_1 + k}$.

To obtain the limit of $\dfrac{k}{h}$, put h, k each equal to zero in the second member,*

$$\lim_{h \doteq 0} \frac{k}{h} = -\frac{x_1}{y_1}.$$

This step is more fully justified in the next article.

This result agrees with that obtained by elementary geometry. The slope of the radius to the circle $x^2 + y^2 = a^2$ through the point (x_1, y_1) is $\dfrac{y_1}{x_1}$, and the slope of the tangent is the negative reciprocal of that of the radius to the point of tangency, since the two lines are perpendicular.

3. Fundamental theorems concerning limits. The following theorems are useful in the processes of the Calculus.

THEOREM 1. If a variable α approaches zero as a limit, then $k\alpha$ will also approach zero, k being any finite constant.

That is, if $\alpha \doteq 0$,

then $k\alpha \doteq 0$.

For, let c be any assigned number. By hypothesis, α can become less than $\dfrac{c}{k}$, hence $k\alpha$ can become less than c, the arbi-

* For convenience, the symbol $\doteq$ will be used to indicate that a variable approaches a constant as a limit; thus the symbolic form $x \doteq a$ is to be read "the variable x approaches the constant a as a limit."

trary, assigned number, hence $k\alpha$ approaches zero as a limit. (Definition of a limit.)

THEOREM 2. Given any finite number n of variables α, β, γ, $\cdots$, each of which approaches zero as a limit, then their sum will approach zero as a limit. For the sum of the n variables does not at any stage numerically exceed n times the largest of them, which by Theorem 1 approaches zero.

THEOREM 3. If each of two variables approaches zero as a limit, their product will approach zero as a limit. More generally, if one variable approaches zero as a limit, then its product with any other variable having a finite limit will have the limit zero, by Theorem 1.

THEOREM 4. If the sum of a finite number of variables is variable, then the limit of their sum is equal to the sum of their limits; *i.e.*,

$$\lim (x + y + \cdots) = \lim x + \lim y + \cdots.$$

For, if $\qquad x \doteq a, \qquad y \doteq b, \cdots,$

then $\qquad x = a + \alpha, \qquad y = b + \beta, \cdots,$

wherein $\qquad \alpha \doteq 0, \qquad \beta \doteq 0, \cdots;$ (Def. of limit)

hence $\qquad x + y + \cdots = (a + b + \cdots) + (\alpha + \beta + \cdots),$

but $\qquad \alpha + \beta + \cdots \doteq 0,$ (Th. 2)

hence, from the definition of a limit,

$$\lim (x + y + \cdots) = a + b + \cdots = \lim x + \lim y + \cdots.$$

THEOREM 5. If the product of a finite number of variables is variable, then the limit of their product is equal to the product of their limits.

For, let $\qquad x = a + \alpha, \qquad y = b + \beta,$

wherein $\qquad \alpha \doteq 0, \qquad \beta \doteq 0,$

so that $\qquad \lim x = a, \qquad \lim y = b.$

Form the product

$$xy = (a+\alpha)(b+\beta) = ab + \alpha b + \beta a + \alpha\beta.$$

Then
$$\lim xy = \lim(ab + \alpha b + \beta a + \alpha\beta)$$
$$= ab + \lim \alpha b + \lim \beta a + \lim \alpha\beta \qquad \text{(Th. 2)}$$
$$= ab. \qquad \text{(Th. 1)}$$

Hence $\lim xy = \lim x \cdot \lim y.$

In the case of a product of three variables x, y, z, we have

$$\lim xyz = \lim xy \cdot \lim z \qquad \text{(Th. 5)}$$
$$= \lim x \lim y \lim z,$$

and so on, for any finite number of variables.

THEOREM 6. If the quotient of two variables x, y is variable, then the limit of their quotient is equal to the quotient of their limits, provided these limits are not both infinite or not both zero.

For, since
$$x = \frac{x}{y}y,$$
$$\lim x = \lim \frac{x}{y} \lim y, \qquad \text{(Th. 5)}$$

and hence
$$\lim \frac{x}{y} = \frac{\lim x}{\lim y}.$$

4. Continuity of functions. When an independent variable x, in passing from a to b, passes through every intermediate value, it is called a *continuous* variable. A function $f(x)$ of an independent variable x is said to be continuous at any value x_1, or in the vicinity of x_1, when $f(x_1)$ is real, finite, and determinate, and such that in whatever way x approaches x_1,

$$\lim_{x \doteq x_1} f(x) = f(x_1).$$

From the definition of a limit it follows that corresponding to a small increment of the variable, the increment of the function is also small, and that corresponding to any number ϵ, previously assigned, another number δ can be determined, such that when h remains numerically less than δ, the difference

$$f(x_1 \pm h) - f(x_1)$$

is numerically less than ϵ.

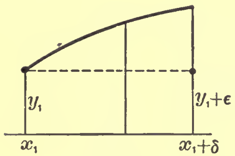

FIG. 3

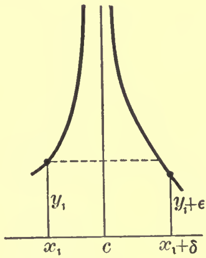

FIG. 4

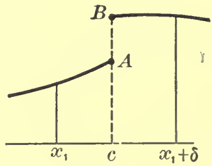

FIG. 5

Thus, the function of Fig. 3 is continuous between the values x_1 and $x_1 + \delta$, while the functions of Fig. 4 and Fig. 5 are discontinuous. In the former of these two the function becomes infinite at $x = c$, while in the latter the difference between the value of the function at $c + h$ and $c - h$ does not approach zero with h, but approaches the finite value AB as h approaches zero.

When a function is continuous for every value of x between a and b, it is said to be continuous within the interval from a to b.

5. Comparison of simultaneous increments of two related variables. The illustrations of Art. 2 suggest the following general procedure for comparing the changes of two related variables. Starting from any fixed pair of values x_1, y_1 represented graphically by the abscissa and ordinate of a chosen point P on a given curve whose equation is given, we change the values of

x and y by the addition of small amounts h and k respectively, so chosen that the new values x_1+h and y_1+k shall be the coördinates of a point P_2 on the curve. The amount h added to x_1 is called the *increment* of x and is entirely arbitrary. Likewise, k is called the increment of y; it is not arbitrary but depends upon the value of h; its value can be calculated when the equation of the curve is given, as is shown by equation (1). These increments are not necessarily positive. In the case of continuous functions, h may always be taken positive. The sign of k will then depend upon the function under consideration. The slope of the line P_1P_2 is then $\frac{k}{h}$ and the slope of the tangent line at P_1 is the limit of $\frac{k}{h}$ as h and consequently k approach zero.

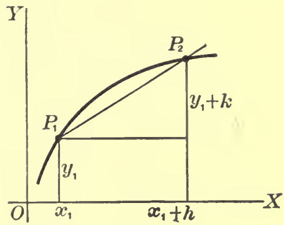

FIG. 6

The determination of the limit of the ratio of k to h as h and k approach zero is the fundamental problem of the Differential Calculus. The process is systematized in the following articles. While the related variables are here represented by ordinate and abscissa of a curve, they may be any two related magnitudes, such as space and time, or volume and pressure of a gas, etc.

6. Definition of a derivative. If to a variable a small increment is given, and if the corresponding increment of a continuous function of the variable is determined, then the limit of the ratio of the increment of the function to the increment of the variable, when the latter increment approaches the limit zero, is called the *derivative* of the function as to the variable.

That is, the derivative is the limit of $\frac{k}{h}$ as h approaches zero,

or
$$\lim_{h \doteq 0}\left(\frac{k}{h}\right).$$

For the purpose of obtaining a derivative in a given case it is convenient to express the process in terms of the following steps:

1. Give a small increment to the variable.

2. Compute the resulting increment of the function.

3. Divide the increment of the function by the increment of the variable.

4. Obtain the limit of this quotient as the increment of the variable approaches zero.

7. Process of differentiation. In the preceding illustrations, the fixed values of x and of y have been written with subscripts to show that only the increments h, k vary during the algebraic process of finding the derivative, also to emphasize the fact that the limit of the ratio of the simultaneous increments h, k depends upon the particular values which the variables x, y have, when they are supposed to take these increments. With this understanding the subscripts will henceforth be omitted. Moreover, the increments h, k will, for greater distinctness, be denoted by the symbols Δx, Δy, read "increment of x," "increment of y."

If the four steps of Art. 6 are applied to the function $y = \phi(x)$, the results become

$$y + \Delta y = \phi(x + \Delta x),$$

$$\Delta y = \phi(x + \Delta x) - \phi(x) = \Delta\phi(x),$$

$$\frac{\Delta y}{\Delta x} = \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} = \frac{\Delta\phi(x)}{\Delta x},$$

$$\lim \frac{\Delta y}{\Delta x} = \lim\left\{\frac{\phi(x + \Delta x) - \phi(x)}{\Delta x}\right\} = \lim \frac{\Delta\phi(x)}{\Delta x}.$$

The operation here indicated is for brevity denoted by the symbol $\frac{d}{dx}$, and the resulting derivative function by $\phi'(x)$; thus

$$\frac{dy}{dx}=\frac{d\phi(x)}{dx}=\lim_{\Delta x \doteq 0}\left\{\frac{\phi(x+\Delta x)-\phi(x)}{\Delta x}\right\}=\phi'(x).$$

The new symbol $\frac{dy}{dx}$ is not (at the present stage at least) to be looked upon as a quotient of two numbers dy, dx, but rather as a single symbol used for the sake of brevity in place of the expression "derivative of y with regard to x."

The process of performing this indicated operation is called the *differentiation* of $\phi(x)$ with regard to x.

EXERCISES

Find the derivatives of the following functions with regard to x.

1. $x^2 - 2x$; $2x$; 3; x.

2. $3x^2 - 4x + 3$.

3. $\frac{1}{4x}$.

4. $x^4 - 2 + \frac{3}{x^2}$.

5. $\frac{1}{x^3}$.

6. x^n, n being a positive integer.

7. $\frac{x^2}{x+1}$.

8. $\frac{x}{x^2+1}$.

9. $y = \sqrt{x}$. [Put $y^2 = x$, and apply the rules.]

10. $y = x^{-\frac{2}{3}}$.

8. Differentiation of a function of a function. Suppose that y, instead of being given directly as a function of x, is expressed as a function of another variable u, which is itself expressed as a function of x. Let it be required to find the derivative of y with regard to the independent variable x.

Let $y=f(u)$, in which u is a function of x. When x changes to the value $x+\Delta x$, let u and y, under the given relations, change to the values $u + \Delta u$, $y + \Delta y$. Then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x};$$

hence, equating limits (Th. 5, Art. 3),

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}.$$

This result may be stated as follows:

The derivative of a function of u with regard to x is equal to the product of the derivative of the function with regard to u, and the derivative of u with regard to x.

EXERCISES

1. Given $y = 3u^2 - 1$, $u = 3x^2 + 4$; find $\frac{dy}{dx}$.

$$\frac{dy}{du} = 6u, \quad \frac{du}{dx} = 6x;$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 36ux = 36x(3x^2 + 4).$$

2. Given $y = 3u^2 - 4u + 5$, $u = 2x^3 - 5$; find $\frac{dy}{dx}$.

3. Given $y = \frac{1}{u}$, $u = 5x^2 - 2x + 4$; find $\frac{dy}{dx}$.

4. Given $y = 3u^2 + \frac{1}{3u^2}$, $u = \frac{x^3}{3} + \frac{3}{x^3}$; find $\frac{dy}{dx}$.

CHAPTER II

DIFFERENTIATION OF THE ELEMENTARY FORMS

In recent articles, the meaning of the symbol $\frac{dy}{dx}$ was explained and illustrated; and a method of expressing its value, as a function of x, was exemplified, in cases in which y was a simple algebraic function of x, by direct use of the definition. This method is not always the most convenient one in the differentiation of more complicated functions.

The present chapter will be devoted to the establishment of some general rules of differentiation which will, in many cases, save the trouble of going back to the definition.

The next five articles treat of the differentiation of algebraic functions and of algebraic combinations of other differentiable functions.

9. Differentiation of the product of a constant and a variable.

Let $$y = cu,$$

Then $$y + \Delta y = c(u + \Delta u),$$

$$\Delta y = c(u + \Delta u) - cu = c\Delta u,$$

$$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x},$$

therefore $$\frac{dy}{dx} = c\frac{du}{dx}.$$

Thus $$\boldsymbol{\frac{d(cu)}{dx} = c\frac{du}{dx}}. \qquad (1)$$

The derivative of the product of a constant and a variable is equal to the constant multiplied by the derivative of the variable.

10. Differentiation of a sum.

Let $$y = u + v - w + \cdots$$

in which $u, v, w, \cdots$ are functions of x.

Then $$y + \Delta y = u + \Delta u + v + \Delta v - w - \Delta w + \cdots,$$

$$\Delta y = \Delta u + \Delta v - \Delta w + \cdots,$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} - \frac{\Delta w}{\Delta x} + \cdots,$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx} + \cdots.$$

Hence $$\frac{d}{dx}(u + v - w + \cdots) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx} + \cdots. \quad (2)$$

The derivative of the sum of a finite number of fractions is equal to the sum of their derivatives.

Cor. If $y = u + c$, c being a constant, then

$$y + \Delta y = u + \Delta u + c;$$

hence $$\Delta y = \Delta u,$$

and $$\frac{dy}{dx} = \frac{du}{dx}.$$

This last equation asserts that all functions which differ from each other only by an additive constant have the same derivative.

Geometrically, the addition of a constant has the effect of moving the curve $y = u(x)$ parallel to the y-axis; this operation will obviously not change the slope at points that have the same x.

From (2), $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dc}{dx};$$

but from the fourth equation above,

$$\frac{dy}{dx} = \frac{du}{dx};$$

hence, it follows that $\quad \dfrac{dc}{dx} = 0.$

The derivative of a constant is zero.

If the number of functions is infinite, Theorem 4 of Art. 3 may not apply; that is, the limit of the sum may not be equal to the sum of the limits, and hence the derivative of the sum may not be equal to the sum of the derivatives. Thus the derivative of an infinite series cannot always be found by differentiating it term by term.

11. Differentiation of a product.

Let $\qquad y = uv$, wherein u, v are both functions of x.

Then $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{(u + \Delta u)(v + \Delta v) - uv}{\Delta x} = u\dfrac{\Delta v}{\Delta x} + v\dfrac{\Delta u}{\Delta x} + \dfrac{\Delta u}{\Delta x} \cdot \Delta v.$

Now let Δx approach zero, using Art. 3, Theorems 4, 5, and noting that if $\dfrac{\Delta u}{\Delta x}$ has a finite limit, then the limit of $\Delta v\left(\dfrac{\Delta u}{\Delta x}\right)$ is zero.

The result may be written in the form

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{3}$$

The derivative of the product of two functions is equal to the sum of the products of the first factor by the derivative of the second, and the second factor by the derivative of the first.

This rule for differentiating a product of two functions may be stated thus: Differentiate the product, treating the first factor as constant, then treating the second factor as constant, and add the two results.

Cor. To find the derivative of the product of three functions uvw.

Let $$y = uvw.$$

By (**3**), $$\frac{dy}{dx} = w\frac{d}{dx}(uv) + uv\frac{dw}{dx}$$

$$= w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + uv\frac{dw}{dx}.$$

The result may be written in the form

$$\frac{d(uvw)}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}. \tag{4}$$

By induction the following rule is at once derived:

The derivative of the product of any finite number of factors is equal to the sum of the products obtained by multiplying the derivative of each factor by all the other factors.

12. Differentiation of a quotient.

Let $y = \frac{u}{v}$, u, v both being functions of x.

Then $$\frac{\Delta y}{\Delta x} = \frac{\frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v(v + \Delta v)}.$$

Passing to the limit, we obtain the result

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}. \tag{5}$$

The derivative of a fraction, the quotient of two functions, is equal to the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, divided by the square of the denominator.

13. Differentiation of a commensurable power of a function.

Let $y = u^n$, in which u is a function of x. Then there are three cases to consider:

1. n a positive integer.
2. n a negative integer.
3. n a commensurable fraction.

1. n a positive integer.

This is a particular case of (4), the factors $u, v, w, \cdots$ all being equal. Thus

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}.$$

2. n a negative integer.

Let $n = -m$, in which m is a positive integer.

Then $$y = u^n = u^{-m} = \frac{1}{u^m},$$

and $$\frac{dy}{dx} = \frac{-mu^{m-1}}{u^{2m}} \cdot \frac{du}{dx} \qquad \text{by (5), and Case (1)}$$

$$= -mu^{-m-1}\frac{du}{dx};$$

hence $$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}.$$

3. n a commensurable fraction.

Let $n = \frac{p}{q}$, where p, q are both integers, which may be either positive or negative.

Then $$y = u^n = u^{\frac{p}{q}};$$

hence $$y^q = u^p,$$

and $$\frac{d}{dx}(y^q) = \frac{d}{dx}(u^p);$$

i.e. $$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}.$$

Solving for the required derivative, we obtain

$$\frac{dy}{dx}=\frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx};$$

hence
$$\boldsymbol{\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}}. \tag{6}$$

The derivative of any commensurable power of a function is equal to the exponent of the power multiplied by the power with its exponent diminished by unity, multiplied by the derivative of the function.

It should be noticed that $\sqrt{u}=u^{\frac{1}{2}}$,

$$\frac{1}{u}=u^{-1},$$

hence
$$\frac{d}{dx}\sqrt{u}=\frac{1}{2\sqrt{u}}\frac{du}{dx}, \qquad \frac{d}{dx}\left(\frac{1}{u}\right)=\frac{-1}{u^2}\frac{du}{dx}.$$

These theorems will be found sufficient for the differentiation of any function that involves only the operations of addition, subtraction, multiplication, division, and involution in which the exponent is an integer or commensurable fraction.

The following examples will serve to illustrate the theorems, and will show the combined application of the general forms (**1**) to (**6**).

ILLUSTRATIVE EXAMPLES

1. $y=\frac{3x^2-2}{x+1}$; find $\frac{dy}{dx}$.

$$\frac{dy}{dx}=\frac{(x+1)\frac{d}{dx}(3x^2-2)-(3x^2-2)\frac{d}{dx}(x+1)}{(x+1)^2}. \quad \text{(by } \mathbf{5}\text{)}$$

$$\frac{d}{dx}(3x^2-2)=\frac{d}{dx}(3x^2)-\frac{d}{dx}(2) \quad \text{(by } \mathbf{2}\text{)}$$

$$=6x. \quad \text{(by } \mathbf{1, 6}\text{)}$$

$$\frac{d}{dx}(x+1)=\frac{dx}{dx}=1. \quad \text{(by } \mathbf{2}\text{)}$$

Substitute these results in the expression for $\frac{dy}{dx}$. Then

$$\frac{dy}{dx} = \frac{(x+1)6x - (3x^2-2)}{(x+1)^2} = \frac{3x^2+6x+2}{(x+1)^2}.$$

2. $u = (3s^2+2)\sqrt{1+5s^2}$; find $\frac{du}{ds}$.

$$\frac{du}{ds} = (3s^2+2)\frac{d}{ds}\sqrt{1+5s^2} + \sqrt{1+5s^2}\cdot\frac{d}{ds}(3s^2+2). \quad \text{(by } \mathbf{3}\text{)}$$

$$\frac{d}{ds}\sqrt{1+5s^2} = \frac{d}{ds}(1+5s^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}(1+5s^2)^{-\frac{1}{2}}\frac{d}{ds}(1+5s^2) \quad \text{(by } \mathbf{6}\text{)}$$

$$= \frac{5s}{\sqrt{1+5s^2}}.$$

$$\frac{d}{ds}(3s^2+2) = 6s. \quad \text{(by } \mathbf{6}\text{)}$$

Substitute these values in the expression for $\frac{du}{ds}$. Then

$$\frac{du}{ds} = \frac{5s(3s^2+2)}{\sqrt{1+5s^2}} + 6s\sqrt{1+5s^2} = \frac{45s^3+16s}{\sqrt{1+5s^2}}.$$

3. $y = \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$; find $\frac{dy}{dx}$.

First, as a quotient,

$$\frac{dy}{dx} = \frac{(\sqrt{1+x^2}-\sqrt{1-x^2})\frac{d}{dx}(\sqrt{1+x^2}+\sqrt{1-x^2})}{(\sqrt{1+x^2}-\sqrt{1-x^2})^2}$$

$$- \frac{(\sqrt{1+x^2}+\sqrt{1-x^2})\frac{d}{dx}(\sqrt{1+x^2}-\sqrt{1-x^2})}{(\sqrt{1+x^2}-\sqrt{1-x^2})^2}. \quad \text{(by } \mathbf{5}\text{)}$$

$$\frac{d}{dx}(\sqrt{1+x^2}+\sqrt{1-x^2}) = \frac{d}{dx}\sqrt{1+x^2} + \frac{d}{dx}\sqrt{1-x^2}. \quad \text{(by } \mathbf{2}\text{)}$$

$$\frac{d}{dx}\sqrt{1+x^2} = \frac{d}{dx}(1+x^2)^{\frac{1}{2}} = \frac{1}{2}(1+x^2)^{-\frac{1}{2}}\frac{d}{dx}(1+x^2). \quad \text{(by } \mathbf{6}\text{)}$$

$$\frac{d}{dx}(1+x^2) = 2x. \quad \text{(by } \mathbf{2} \text{ and } \mathbf{6}\text{)}$$

Similarly for the other terms. Combining the results, we have

$$\frac{dy}{dx} = \frac{-2}{x^3}\left(1 + \frac{1}{\sqrt{1-x^4}}\right).$$

Ex. 3 may also be worked by first rationalizing the denominator.

EXERCISES

Find the x-derivatives of the following functions:

1. $y = x^{10}$.

2. $y = x^{-8}$.

3. $y = c\sqrt{x}$.

4. $y = \dfrac{2}{\sqrt{x}} - \dfrac{\sqrt[3]{x}}{3}$.

5. $y = \sqrt[4]{x^{-5}}$.

6. $y = (x+a)^n$.

7. $y = x^n + a^n$.

8. $y = \dfrac{x}{\sqrt{a^2 - x^2}}$.

9. $y = \dfrac{x+3}{x^2+2}$.

10. $y = (x+1)\sqrt{x+2}$.

11. $y = \dfrac{\sqrt{a+x}}{\sqrt{a} + \sqrt{x}}$.

12. $y = \sqrt{\dfrac{1+x}{1-x}}$.

13. $y = \dfrac{x}{x + \sqrt{1-x^2}}$.

14. $y = (2a^{\frac{1}{2}} + x^{\frac{1}{2}})\sqrt{a^{\frac{1}{2}} + x^{\frac{1}{2}}}$.

15. $y = \left\{\dfrac{x}{1+\sqrt{1-x^2}}\right\}^n$.

16. $y = \sqrt{\dfrac{1-x^2}{(1+x^2)^3}}$.

17. $y = \dfrac{x^n + 1}{x^n - 1}$.

18. $y = \dfrac{1}{(a+x)^m} \cdot \dfrac{1}{(b+x)^n}$.

19. $y = \dfrac{3x^3 + 2}{x(x^3+1)^{\frac{2}{3}}}$.

20. $y = 3(x^2+1)^{\frac{4}{3}}(4x^2 - 3)$.

21. $y = 3u^2 - 7$.

22. $y = 4u^3 - 6u^2 + 12u - 3$.

23. $y = (1 - 3u^2 + 6u^4)(1+u^2)^3$.

24. $y = ux$.

25. $y = u^2 + 3xu^2 + x^4$.

26. $y = \dfrac{u^n}{(a+x)^n}$.

27. $y = u^2x^3w$.

28. Given $(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$; find $(a+x)^4$ by differentiation.

29. Show that the slope of the tangent to the curve $y = x^3$ is never negative. Show where the slope increases or decreases.

30. Given $b^2x^2 + a^2y^2 = a^2b^2$, find $\frac{dy}{dx}$: (1) by differentiating as to x; (2) by differentiating as to y; (3) by solving for y and differentiating as to x. Compare the results of the three methods.

31. Show that form **(1)**, p. 25, is a special case of **(3)**, p. 27.

32. At what point of the curve $y^2 = ax^3$ is the slope 0? -1? $+1$?

33. Trace the curve $y = x^3 + 3x^2 + x - 1$.

34. $y = \frac{3u^2 + 7}{\sqrt{7u^2 + 5}}$ and $u = 5x^2 - 1$; find $\frac{dy}{dx}$.

35. At what angle do the curves $y^2 = 12x$ and $y^2 + x^2 + 6x - 63 = 0$ intersect?

14. Differentiation of implicit functions. When a functional relation between x and y cannot be readily solved for y, the preceding rules may be applied directly to the *implicit* function. The derivative will usually contain both x and y. Thus the derivative of an *algebraic* function, defined by equating a polynomial in x and y to zero, may be obtained by the process illustrated in the following examples:

Ex. **1.** Given the function y of x, defined by the equation

$$x^5 + y^5 - 5xy + 1 = 0,$$

find $\frac{dy}{dx}$.

Since $$\frac{d}{dx}(x^5 + y^5 - 5xy + 1) = 0,$$

hence $$5x^4 + 5y^4\frac{dy}{dx} - 5y - 5x\frac{dy}{dx} = 0, \quad \text{(by } \mathbf{2, 3}\text{)}$$

Solving for $\frac{dy}{dx}$, we obtain $$\frac{dy}{dx} = \frac{x^4 - y}{x - y^4}.$$

Ex. **2.** $xy^2 + x^2y = 1$. Find $\frac{dy}{dx}$.

Ex. **3.** $x + y + (x - y)^2 + (2x - 3y)^3 = 0$. Find $\frac{dy}{dx}$.

15. Elementary transcendental functions. The following functions are called *transcendental* functions:

Simple exponential functions, consisting of a constant number raised to a power whose exponent is variable, as 4^x, a^x;

the *logarithmic* functions, as $\log_a x$, $\log_b u$;

the *incommensurable powers* of a variable, as $x^{\sqrt{2}}$, u^π;

the *trigonometric* functions, as $\sin u$, $\cos u$;

the *inverse trigonometric* functions, as $\sin^{-1} u$, $\tan^{-1} x$.

There are still other transcendental functions, but they will not be considered in this book.

The next four articles treat of the logarithmic, the two exponential functions, and the incommensurable power.

16. Differentiation of $\log_a x$ and $\log_a u$.

Let $$y = \log_a x.$$

Then $$y + \Delta y = \log_a (x + \Delta x),$$

$$\frac{\Delta y}{\Delta x} = \frac{\log_a (x + \Delta x) - \log_a x}{\Delta x}$$

$$= \frac{1}{\Delta x} \log_a \left(\frac{x + \Delta x}{x}\right).$$

For convenience writing h for Δx, and rearranging, we obtain

$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \cdot \frac{x}{h} \log_a \left(1 + \frac{h}{x}\right)$$

$$= \frac{1}{x} \log_a \left(1 + \frac{h}{x}\right)^{\frac{x}{h}};$$

whence $$\frac{dy}{dx} = \frac{1}{x} \lim_{h \doteq 0} \left[\log_a \left(1 + \frac{h}{x}\right)^{\frac{x}{h}}\right].$$

To evaluate the expression $\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$ when $h \doteq 0$, expand it by the binomial theorem, supposing $\frac{x}{h}$ to be a positive integer m.

The expansion may be written

$$\left(1+\frac{1}{m}\right)^m = 1+m\cdot\frac{1}{m}+\frac{m(m-1)}{1\cdot 2}\cdot\frac{1}{m^2}+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{m^3}+\cdots,$$

which can be put in the form

$$\left(1+\frac{1}{m}\right)^m = 1+1+\frac{1}{1}\frac{\left(1-\frac{1}{m}\right)}{2}+\frac{1}{1}\frac{\left(1-\frac{1}{m}\right)}{2}\frac{\left(1-\frac{2}{m}\right)}{3}+\cdots.$$

Now as m becomes very large, the terms $\frac{1}{m}, \frac{2}{m}, \cdots$ become very small and m increases without limit as h approaches zero. As $m \doteq \infty$ the series approaches the limit

$$1+1+\frac{1}{2\,!}+\frac{1}{3\,!}+\frac{1}{4\,!}+\cdots,$$

which will be discussed later.

The numerical value of this limit can be readily calculated to any desired approximation. This number is an important constant, which is denoted by the letter e, and is equal to $2.7182818\cdots$; thus

$$\lim_{m \doteq \infty}\left(1+\frac{1}{m}\right)^m = e = 2.7182818\cdots.^*$$

* This method of obtaining e is rather too brief to be rigorous; it assumes that $\frac{x}{\Delta x}$ is a positive integer, but that is equivalent to restricting Δx to approach zero in a particular way. It also applies the theorems of limits to the sum and product of an infinite number of terms. The proof is completed on p. 315 of McMahon and Snyder's "Differential Calculus."

The number e is known as the *natural* or Naperian *base;* and logarithms to this base are called natural or Naperian logarithms. Natural logarithms will be written without a subscript, as $\log x$; for other bases a subscript, as in $\log_a x$, will generally be used to designate the base. The logarithm of e to any base a is called the *modulus* of the system whose base is a.

If the value $\lim_{h \doteq 0}\left(1+\frac{h}{x}\right)^{\frac{x}{h}} = e$ is substituted in the expression for $\frac{dy}{dx}$, the result is

$$\frac{dy}{dx} = \frac{1}{x} \cdot \log_a e.$$

More generally, by Art. 8,

$$\frac{d}{dx} \log_a u = \frac{\log_a e}{u} \frac{du}{dx}. \tag{7}$$

In the particular case in which $a = e$,

$$\frac{d}{dx} \log u = \frac{1}{u} \frac{du}{dx}. \tag{8}$$

The derivative of the logarithm of a function is the product of the derivative of the function and the modulus of the system of logarithms, divided by the function.

17. Differentiation of the simple exponential function.

Let $\qquad y = a^u.$

Then $\qquad \log y = u \log a.$

Differentiating both members of this identity as to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \log a \cdot \frac{du}{dx}, \quad \text{(by 8)},$$

$$\frac{dy}{dx} = \log a \cdot y \cdot \frac{du}{dx};$$

therefore
$$\frac{d}{dx} a^u = \log a \cdot a^u \cdot \frac{du}{dx}. \tag{9}$$

In the particular case in which $a = e$,

$$\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}. \tag{10}$$

The derivative of an exponential function with a constant base is equal to the product of the function, the natural logarithm of the base, and the derivative of the exponent.

18. Differentiation of an incommensurable power.

Let
$$y = u^n,$$
in which n is an incommensurable constant. Then

$$\log y = n \log u,$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{u} \cdot \frac{du}{dx},$$

$$\frac{dy}{dx} = n \cdot \frac{y}{u} \cdot \frac{du}{dx},$$

$$\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}.$$

This has the same form as (6), so that the qualifying word "commensurable" of Art. 13 can now be omitted.

EXERCISES

Find the x derivatives of the following functions:

1. $y = \log(x + a)$.

2. $y = \log(ax + b)$.

3. $y = \log(4x^2 - 7x + 2)$.

4. $y = \log \dfrac{1 + x}{1 - x}$.

5. $y = \log \dfrac{1 + x^2}{1 - x^2}$.

6. $y = x \log x$.

7. $y = x^n \log x$.

8. $y = x^n \log x^m$.

9. $y = \log \sqrt{1 - x^2}$.

10. $y = \sqrt{x} - \log(\sqrt{x} + 1)$.

11. $y = \log_a(3x^2 - \sqrt{2 + x})$.

12. $y = \log_{10}(x^2 + 7x)$.

13. $y = \log_x a$.

14. $y = e^{xa}$.

15. $y = e^{4x+5}$.

16. $y = e^{\frac{1}{1+x}}$.

17. $y = \dfrac{e^x}{1+e^x}$.

18. $y = e^x(1-x^3)$.

19. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

20. $y = \log(e^x - e^{-x})$.

21. $y = \log(x + e^x)$.

22. $y = x^n a^x$.

23. $y = \log \dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$.

24. $y = \dfrac{1}{\log x}$.

25. $y = (\log x)^2$.

26. $y = \log(\log x)$.

27. $y = x \log \dfrac{1}{x}$.

28. $y = a^{\log x}$.

The following functions can be easily differentiated by first taking the logarithms of both members of the equations.

29. $y = \dfrac{(x-1)^{\frac{5}{2}}}{(x-2)^{\frac{3}{4}}(x-3)^{\frac{7}{3}}}$.

30. $y = x\sqrt{1-x}(1+x)$.

31. $y = \dfrac{x(1+x^2)}{\sqrt{1-x^2}}$.

32. $y = x^5(a+3x)^3(a-2x)^2$.

33. $y = \dfrac{\sqrt{(x+a)^3}}{\sqrt{x-a}}$.

19. Limit of $\dfrac{\sin\theta}{\theta}$ as θ approaches 0. Before proceeding to determine the derivatives of the trigonometric functions it is necessary to prove the following lemma:

$$\lim_{\theta \doteq 0} \frac{\sin\theta}{\theta} = 1.$$

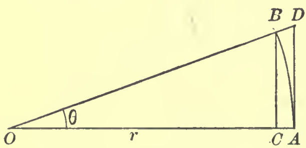

FIG. 7

With O as a center and $OA = r$ as radius, describe the circular arc AB. Let the tangent at A meet OB produced in D; draw BC perpendicular to OA, cutting OA in C. Let the angle $OAB = \theta$ in radian measure,

then $\qquad \text{arc } AB = r\theta,$

$$CB < \text{arc } AB < AD, \qquad \text{by geometry}$$

i.e. $\qquad r\sin\theta < r\theta < r\tan\theta,$

$$\sin\theta < \theta < \tan\theta.$$

By dividing each member of these inequalities by $\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \sec\theta;$$

but $\sec\theta = 1$, when $\theta = 0$,

hence, $$\lim_{\theta \doteq 0} \frac{\theta}{\sin\theta} = 1, \text{ and } \lim_{\theta \doteq 0} \frac{\sin\theta}{\theta} = 1.$$

20. Differentiation of sin *u*.

Let $$y = \sin u.$$

Then $$\frac{\Delta y}{\Delta x} = \frac{\sin(u + \Delta u) - \sin u}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

To evaluate the expression

$$\sin(u + \Delta u) - \sin u,$$

we make use of the formulas for the sine of the sum and the sine of the difference of two angles. Since

$$\sin(a + b) = \sin a \cos b + \cos a \sin b,$$
$$\sin(a - b) = \sin a \cos b - \cos a \sin b,$$

hence, by subtracting the second equation from the first,

$$\sin(a + b) - \sin(a - b) = 2 \cos a \sin b.$$

This equation is true for all values of a and of b. In particular, then, putting

$$a + b = u + \Delta u,$$

and $$a - b = u,$$

that is, $$a = u + \frac{\Delta u}{2}, \text{ and } b = \frac{\Delta u}{2},$$

we obtain

$$\sin(u + \Delta u) - \sin u = 2 \cos\left(u + \frac{\Delta u}{2}\right) \sin\frac{\Delta u}{2}.$$

The expression for $\frac{\Delta y}{\Delta x}$ may now be written in the form

$$\frac{\Delta y}{\Delta x} = \cos\left(u + \frac{\Delta u}{2}\right) \cdot \frac{\sin \frac{\Delta u}{2}}{\frac{\Delta u}{2}} \frac{\Delta u}{\Delta x},$$

hence
$$\frac{dy}{dx} = \cos u \cdot \lim_{\Delta u \doteq 0} \left[\frac{\sin \frac{\Delta u}{2}}{\frac{\Delta u}{2}}\right] \frac{du}{dx},$$

hence, by Art. 19,
$$\boldsymbol{\frac{d}{dx} \sin u = \cos u \frac{du}{dx}}. \qquad \textbf{(11)}$$

The derivative of the sine of a function is equal to the product of the cosine of the function and the derivative of the function.

21. Differentiation of cos u.

Let
$$y = \cos u = \sin\left(\frac{\pi}{2} - u\right).$$

Then
$$\frac{dy}{dx} = \frac{d}{dx} \sin\left(\frac{\pi}{2} - u\right) = \cos\left(\frac{\pi}{2} - u\right) \frac{d}{dx}\left(\frac{\pi}{2} - u\right),$$

$$\boldsymbol{\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}}. \qquad \textbf{(12)}$$

The derivative of the cosine of a function is equal to minus the product of the sine of the function and the derivative of the function.

22. Differentiation of tan u.

Let
$$y = \tan u = \frac{\sin u}{\cos u}.$$

Then
$$\frac{dy}{dx} = \frac{\cos u \cdot \frac{d}{dx} \sin u - \sin u \cdot \frac{d}{dx} \cos u}{\cos^2 u} \qquad \text{(by } \mathbf{5}\text{)}$$

$$= \frac{\cos^2 u \cdot \frac{du}{dx} + \sin^2 u \cdot \frac{du}{dx}}{\cos^2 u} = \frac{\frac{du}{dx}}{\cos^2 u}, \qquad \text{(by } \mathbf{11, 12}\text{)}$$

that is,
$$\boldsymbol{\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}}. \qquad \textbf{(13)}$$

The derivative of the tangent of a function is equal to the product of the square of the secant of the function and the derivative of the function.

Since the remaining elementary trigonometric functions can be expressed as rational functions of those already considered, their derivatives can be obtained by means of the preceding rules. The results are

$$\frac{d}{dx}\cot u = -\csc^2 u\,\frac{du}{dx}. \tag{14}$$

$$\frac{d}{dx}\sec u = \sec u \tan u\,\frac{du}{dx}. \tag{15}$$

$$\frac{d}{dx}\csc u = -\csc u \cot u\,\frac{du}{dx}. \tag{16}$$

EXERCISES

Find the x-derivatives of the following functions:

1. $y = \sin 7x$.
2. $y = \cos 5x$
3. $y = \sin x^2$.
4. $y = \sin 2x \cos x$.
5. $y = \sin^3 x$.
6. $y = \sin 5x^2$.
7. $y = \sin^2 7x$.
8. $y = \frac{1}{3}\tan^3 x - \tan x$.
9. $y = \sin^3 x \cos x$.
10. $y = \tan x + \sec x$.
11. $y = \sin^2 (1 - 2x^2)^2$.
12. $y = \tan (3 - 5x^2)^2$.
13. $y = \tan^2 x - \log(\sec^2 x)$.
14. $y = \log \tan (\frac{1}{2}x + \frac{1}{4}\pi)$.
15. $y = \log \sin \sqrt{x}$.
16. $y = \tan a^{\frac{1}{x}}$.
17. $y = \sin nx \sin^n x$.
18. $y = \sin (u + b) \cos (u - b)$.
19. $y = \dfrac{\sin^m nx}{\cos^n mx}$.
20. $y = x + \log\cos\left(x - \dfrac{\pi}{4}\right)$.
21. $y = \sin(\sin u)$.
22. $y = \sin^2 e^{ax}$.
23. $y = \sin e^x \cdot \log x$.
24. $y = \sqrt{\sin x^2}$.
25. $y = \csc^2 4x$.
26. $y = \sec(4x - 3)^2$.
27. $y = \cot x^2 + \sec\sqrt{x}$.
28. $y = \sin xy$.
29. $y = \tan(x + y)$.

30. Find $\frac{d}{dx}(\cos u)$ directly from the definition of the derivative. Also $\frac{d}{dx}(\tan u)$.

31. Find $\frac{d}{dx}(\cos u)$ from the relation $\sin^2 u + \cos^2 u = 1$.

23. Differentiation of $\sin^{-1} u$.

Let $$y = \sin^{-1} u.$$

Then $$\sin y = u,$$

and, by differentiating both members of this identity,

$$\cos y \frac{dy}{dx} = \frac{du}{dx};$$

hence $$\frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx} = \frac{1}{\pm\sqrt{1-\sin^2 y}}\frac{du}{dx};$$

i.e. $$\frac{d}{dx}\sin^{-1} u = \pm\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}.$$

The ambiguity of sign accords with the fact that $\sin^{-1} u$ is a many-valued function of u, since, for any value of u between -1 and 1, there is a series of angles whose sine is u: and, when u receives an increase, some of these angles increase and some decrease; hence, for some of them, $\frac{d \sin^{-1} u}{du}$ is positive, and for some negative. It will be seen that, when $\sin^{-1} u$ lies in the first or fourth quarter, it increases with u, and, when in the second or third, it decreases as u increases. Hence, for the angles of the first and fourth quarters,

$$\frac{d}{dx}\sin^{-1} u = -\frac{d}{dx}\cos^{-1} u = +\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}. \qquad (17)$$

In the other quarters the minus sign is to be used before the radical.

The derivatives of the other inverse trigonometric functions can be easily obtained by the method employed in the present article. The most important of the remaining ones are $\tan^{-1} u$, $\sec^{-1} u$;

$$\frac{d}{dx}\tan^{-1} u = -\frac{d}{dx}\cot^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}. \tag{18}$$

$$\frac{d}{dx}\sec^{-1} u = -\frac{d}{dx}\csc^{-1} u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}. \tag{19}$$

EXERCISES

Find the x-derivatives of each of the following functions:

1. $y = \sin^{-1} 2x^2.$
2. $y = \cos^{-1}\sqrt{1-x^2}.$
3. $y = \sin^{-1}(3x-1).$
4. $y = \sin^{-1}(3x-4x^3).$
5. $y = \sin^{-1}\dfrac{1-x^2}{1+x^2}.$
6. $y = \sqrt{\sin^{-1} x}.$
7. $y = \tan^{-1} e^x.$
8. $y = \cos^{-1}\log x.$
9. $y = \sin^{-1}(\tan x).$
10. $y = \sec^{-1}\dfrac{1}{\sqrt{1-x^2}}.$
11. $y = \csc^{-1}\dfrac{1}{x}.$
12. $y = \tan^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right).$
13. $y = \tan^{-1}\dfrac{1}{x}.$
14. $y = \sin^{-1}\sqrt{\sin x}.$
15. $y = \tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}.$
16. $y = \tan x \cdot \tan^{-1} x.$
17. $y = x\sin^{-1} x.$
18. $y = e^{\tan^{-1} x}.$
19. $y = \csc^{-1}\dfrac{1}{2x^2-1}.$
20. $y = \sec^{-1}\dfrac{x^2+1}{x^2-1}.$
21. $y = \tan^{-1}\dfrac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}.$
22. $y = \cos^{-1}\dfrac{e^x-e^{-x}}{e^x+e^{-x}}.$
23. $y = \tan^{-1}(n\tan x).$
24. $y = \cos^{-1}(\cos 2x).$
25. $y = \cos^{-1}(2\cos x).$
26. $y = \tan^{-1}(\sqrt{1+x^2}-x).$
27. $y = 2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}.$
28. $y = \tan^{-1}\dfrac{2x-b}{b\sqrt{3}} + \tan^{-1}\dfrac{2b-x}{x\sqrt{3}}.$

24. Table of fundamental forms.

$$\frac{d(cu)}{dx} = c\frac{du}{dx}. \tag{1}$$

$$\frac{d}{dx}(u+v-w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}. \tag{2}$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{3}$$

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}. \tag{4}$$

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}. \tag{5}$$

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \tag{6}$$

$$\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx}. \tag{7}$$

$$\frac{d}{dx}\log u = \frac{1}{u}\frac{du}{dx}. \tag{8}$$

$$\frac{d}{dx}a^u = \log a \cdot a^u \cdot \frac{du}{dx}. \tag{9}$$

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}. \tag{10}$$

$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}. \tag{11}$$

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}. \tag{12}$$

$$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}. \tag{13}$$

$$\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx} \tag{14}$$

$$\frac{d}{dx}\sec u \quad = \sec u \tan u \frac{du}{dx}. \tag{15}$$

$$\frac{d}{dx}\csc u \quad = -\csc u \cot u \frac{du}{dx}. \tag{16}$$

$$\frac{d}{dx}\sin^{-1} u = -\frac{d}{dx}\cos^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}. \tag{17}$$

$$\frac{d}{dx}\tan^{-1} u = -\frac{d}{dx}\cot^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}. \tag{18}$$

$$\frac{d}{dx}\sec^{-1} u = -\frac{d}{dx}\csc^{-1} u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}. \tag{19}$$

EXERCISES ON CHAPTER II

Find the x-derivatives of the following functions:

1. $y = 3x^2 + 5x^3 - 7.$

2. $y = \frac{3}{x^2} + \frac{5}{x^3} - \frac{1}{7}.$

3. $y = (x+5)\sqrt{x-3}.$

4. $y = x\sqrt{a^2 - x^2}.$

5. $y = x \log \sin x.$

6. $y = \frac{a}{x}\sqrt{a^2 - x^2}.$

7. $y = \frac{c^2}{x} e^{\frac{x}{c}}.$

8. $y = \tan 2z, z = \tan^{-1}(2x - 1).$

9. $y = e^{\sqrt{u}},\ u = \log \sin x.$

10. $y = \log \frac{x}{a^x}.$

11. $y = \frac{1-x^2}{\sqrt{1+x^2}}.$

12. $y = e^x \cos x.$

13. $y = \cos^{-1}\left(\frac{1}{x}\right).$

14. $y = \tan^{-1}\frac{4\sin x}{3 + 5\cos x}$

15. $y = (x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax}.$

16. $y = \cot^{-1}\frac{1+\sqrt{1+x^2}}{x}.$

17. $y = \tan^4 x - 2\tan^2 x + \log(\sec^4 x).$

18. $y = \frac{x\log x}{1-x} + \log(1-x).$

19. $y = \cos^{-1}\frac{3+5\cos x}{5+3\cos x}.$

20. $y = \log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} - \frac{1}{2}\tan^{-1} x.$

21. $y = \log(x + \sqrt{x^2 - a^2}) + \sec^{-1}\frac{x}{a}.$

22. $y = e^u,\ u = \log x.$

23. $y = \log s^2 + e^s,\ s = \sec x.$

24. $x^3 + y^3 - 3\,axy = 0.$

25. $x^2y^2 + x^3 + y^3 = 0.$

26. $x^3 + x = y + y^3.$

27. $xy^2 + x^2y = x + y.$

28. $y = \sin(2\,u - 7),\ u = \log x^2.$

29. By means of differentiation eliminate the constant p from the equation $y = px^2$.

30. At what points is the tangent to the curve $y = \cos x$ parallel to the x-axis?

31. Show that the x-derivative of $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ is not a function of x.

32. Find at what points of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the tangents cut off equal intercepts on the axes.

33. Find the points at which the slope of the curve $y = \tan x$ is twice that of the line $y = x$.

34. Find the angle which the curves $y = \sin x$ and $y = \cos x$ make with each other at their point of intersection.

CHAPTER III

SUCCESSIVE DIFFERENTIATION

25. Definition of *n*th derivative. When a given function $y = \phi(x)$ is differentiated with regard to x by the rules of Chapter I, then the result

$$\frac{dy}{dx} = \phi'(x)$$

is a new function of x which may itself be differentiated by the same rules. Thus,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\phi'(x).$$

The left-hand member is usually abbreviated to $\frac{d^2y}{dx^2}$, and the right-hand member to $\phi''(x)$; that is,

$$\frac{d^2y}{dx^2} = \phi''(x).$$

Differentiating again and using a similar notation, we obtain

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = \phi'''(x),$$

and so on for any number of differentiations. Thus the symbol $\frac{d^2y}{dx^2}$ expresses that y is to be differentiated with regard to x, and that the resulting derivative is then to be differentiated. Similarly, $\frac{d^3y}{dx^3}$ indicates the performance of the operation $\frac{d}{dx}$ three times, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)$. In general, the symbol $\frac{d^ny}{dx^n}$ means that y is to be differentiated n times in succession with regard to x.

Ex. 1. If
$$y = x^4 + \sin 2x,$$
$$\frac{dy}{dx} = 4x^3 + 2\cos 2x,$$
$$\frac{d^2y}{dx^2} = 12x^2 - 4\sin 2x,$$
$$\frac{d^3y}{dx^3} = 24x - 8\cos 2x,$$
$$\frac{d^4y}{dx^4} = 24 + 16\sin 2x.$$

If an implicit equation between x and y is given and the derivatives of y with regard to x are required, it is not necessary to solve the equation for either variable before performing the differentiation.

Ex. 2. Given $x^4 + y^4 + 4a^2xy = 0$; find $\frac{d^2y}{dx^2}$.
$$\frac{d}{dx}(x^4 + y^4 + 4a^2xy) = 0,$$
$$\frac{d}{dx}x^4 + \frac{d}{dx}y^4 + 4a^2\frac{d}{dx}xy = 0,$$
$$4x^3 + 4y^3\frac{dy}{dx} + 4a^2x\frac{dy}{dx} + 4a^2y = 0.$$

The last equation is now to be solved for $\frac{dy}{dx}$,
$$\frac{dy}{dx} = -\frac{x^3 + a^2y}{y^3 + a^2x}. \tag{1}$$
Differentiating again, we obtain
$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x^3 + a^2y}{y^3 + a^2x}\right)$$
$$= -\frac{(y^3 + a^2x)\frac{d}{dx}(x^3 + a^2y) - (x^3 + a^2y)\frac{d}{dx}(y^3 + a^2x)}{(y^3 + a^2x)^2}$$
$$= -\frac{(y^3 + a^2x)\left(3x^2 + a^2\frac{dy}{dx}\right) - (x^3 + a^2y)\left(3y^2\frac{dy}{ax} + a\right)}{(y^3 + a^2x)^2}.$$

The value of $\frac{dy}{dx}$ from (1) is now to be substituted in the last equation, and the resulting expression simplified. The final form may be written:

$$\frac{d^2y}{dx^2}=\frac{2\,a^6xy-10\,a^2x^3y^3-a^4(x^4+y^4)-3\,x^2y^2(x^4+y^4)}{(y^3+a^2x)^3}.$$

In like manner higher derivatives may be found.

26. Expression for the nth derivative in certain cases. For certain functions, a general expression for the nth derivative can be readily obtained in terms of n.

Ex. **1.** If $y=e^x$, then $\frac{dy}{dx}=e^x,\ \frac{d^2y}{dx^2}=e^x,\ \cdots,\ \frac{d^ny}{dx^n}=e^x.$ where n is any positive integer. If $y=e^{ax}$, $\frac{d^ny}{dx^n}=a^ne^{ax}$.

Ex. **2.** If

$$y=\sin x,$$
$$\frac{dy}{dx}=\cos x=\sin\left(x+\frac{\pi}{2}\right),$$
$$\frac{d^2y}{dx^2}=\cos\left(x+\frac{\pi}{2}\right)=\sin\left(x+\frac{2\,\pi}{2}\right),$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{d^ny}{dx^n}=\sin\left(x+\frac{n\pi}{2}\right).$$

If $\quad y=\sin ax,\ \frac{d^ny}{dx^n}=a^n\sin\left(ax+\frac{n\pi}{2}\right).$

EXERCISES ON CHAPTER III

1. $y=3\,x^4+5\,x^2+3\,x-9$; find $\frac{d^3y}{dx^3}$.

2. $y=2\,x^2+3\,x+5$; find $\frac{d^3y}{dx^3}$.

3. $y=\frac{1}{x}$; find $\frac{d^3y}{dx^3}$.

4. $y=x^2-\frac{1}{x^2}$; find $\frac{d^4y}{dx^4}$.

5. $y=\tan x$; find $\frac{d^3y}{dx^3}$.

6. $y=e^x\log x$; find $\frac{d^2y}{dx^2}$.

7. $y=x^2\log x$; find $\frac{d^2y}{dx^2}$.

8. $y=\sec^2 x$; find $\frac{d^3y}{dx^3}$.

9. $y = \log \sin x$; find $\dfrac{d^3y}{dx^3}$.

10. $y = \sin x \cos x$; find $\dfrac{d^4y}{dx^4}$.

11. $y = \dfrac{x^3}{1-x}$; find $\dfrac{d^4y}{dx^4}$.

12. $y = x^4 \log x^2$; find $\dfrac{d^5y}{dx^5}$.

13. $y = \sin x$; find $\dfrac{d^{12}y}{dx^{12}}$.

14. $y = \log(e^x + e^{-x})$; find $\dfrac{d^3y}{dx^3}$.

15. $y = (x^2 - 3x + 3)e^{2x}$; find $\dfrac{d^3y}{dx^3}$.

16. $y = x^4 \log x$; find $\dfrac{d^6y}{dx^6}$.

17. $y = e^{ax}$; find $\dfrac{d^ny}{dx^n}$.

18. $y = \dfrac{1}{x-1}$; find $\dfrac{d^ny}{dx^n}$.

19. $y = \cos mx$; find $\dfrac{d^ny}{dx^n}$.

20. $y = \dfrac{1}{(a+x)^m}$; find $\dfrac{d^ny}{dx^n}$.

21. $y = \log(a+x)^m$; find $\dfrac{d^ny}{dx^n}$.

22. $y^2 = 2px$; find $\dfrac{d^3y}{dx^3}$.

23. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; find $\dfrac{d^2y}{dx^2}$.

24. $x^3 + y^3 = 3axy$; find $\dfrac{d^2y}{dx^2}$.

25. $e^{x+y} = xy$; find $\dfrac{d^2y}{dx^2}$.

26. $y = 1 + xe^y$; find $\dfrac{d^2y}{dx^2}$.

27. $y = e^x \sin x$; prove $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$.

28. $y = ax \sin x$; prove $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + (x^2+2)y = 0$.

29. $y = ax^{n+1} + bx^{-n}$; prove $x^2\dfrac{d^2y}{dx^2} = n(n+1)y$.

30. $y = (\sin^{-1}x)^2$; prove $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2$.

31. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$; prove $\dfrac{dy}{dx} = 1 - y^2$.

32. $y = x^{n-1}\log x$; find $\dfrac{d^ny}{dx^n}$.

33. $y = x^2e^x$; prove $\dfrac{d^ny}{dx^n} = 2\dfrac{d^{n-1}y}{dx^{n-1}} - \dfrac{d^{n-2}y}{dx^{n-2}} + 2e^x$.

34. $y = \cos^2 x$; find $\dfrac{d^ny}{dx^n}$.

CHAPTER IV

MAXIMA AND MINIMA

27. Increasing and decreasing functions. A function is said to be *increasing* if it increases as the variable increases and decreases as the variable decreases. A function is said to be *decreasing* if it decreases as the variable increases and increases as the variable decreases. When the graph of the function is known it will indicate whether the function is increasing or decreasing for an assigned value of x; conversely, a knowledge of the fact whether a function is increasing or decreasing is of great assistance in drawing the graph. Usually a function is increasing for certain values of x and decreasing for others.

28. Test for determining intervals of increasing and decreasing. Let $y = \phi(x)$ be a continuous function having a derivative for all values of x from a to b. By the above definition y is increasing or decreasing at a point x_1, according as

$$k = \phi(x_1 + h) - \phi(x_1)$$

has or has not the same sign as h, where h is a sufficiently small number. Hence $\phi(x)$ is an increasing or a decreasing function at the value x_1 according as

$$\frac{dy}{dx} = \lim_{h \doteq 0} \left\{ \frac{\phi(x_1 + h) - \phi(x_1)}{h} \right\} = \phi'(x_1)$$

is positive or negative.

Thus, the function $y = \phi(x)$ is increasing, if $\phi'(x)$ is positive; if $\phi'(x)$ is negative, the function is decreasing.

In order that a function shall change from an increasing function to a decreasing function or *vice versa*, it is necessary and sufficient that its derivative shall change sign. If the derivative is continuous, this can happen only when the derivative passes through the value zero. The derivative may also change sign when it becomes infinite, and, notwithstanding this discontinuity of the derivative, the original function may still be continuous. In the graph of the function this requires that at such a point the tangent to the locus shall be parallel to the y-axis. The process will be illustrated by a few examples.

Ex. Find the intervals in which the function

$$\phi(x) = 2x^3 - 9x^2 + 12x - 6$$

is increasing or decreasing. The derivative is

$$\phi'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2);$$

hence, as x passes from $-\infty$ to 1, the derived function $\phi'(x)$ is positive and $\phi(x)$ increases from $\phi(-\infty)$ to $\phi(1)$, *i.e.* from $\phi = -\infty$ to $\phi = -1$; as x passes from 1 to 2, $\phi'(x)$ is negative, and $\phi(x)$ decreases from $\phi(1)$ to $\phi(2)$, *i.e.* from -1 to -2; and as x passes from 2 to $+\infty$, $\phi'(x)$ is positive, and $\phi(x)$ increases from $\phi(2)$ to $\phi(\infty)$, *i.e.* from -2 to $+\infty$. The locus of the equation $y = \phi(x)$ is shown in Fig. 8. At points where $\phi'(x) = 0$, the function $\phi(x)$ is neither increasing nor decreasing. At such points the tangent is parallel to the axis of x. Thus in this illustration, at $x = 1$, $x = 2$, the tangent is parallel to the x-axis.

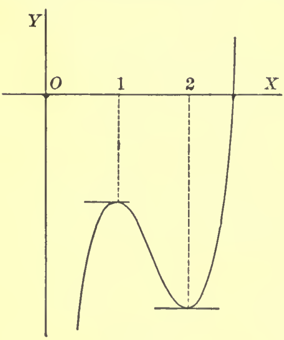

FIG. 8

EXERCISES

1. Find the intervals of increasing and decreasing for the function

$$\phi(x) \equiv x^3 + 2x^2 + x - 4.$$

Here $$\phi'(x) = 3x^2 + 4x + 1 = (3x+1)(x+1).$$

The function increases from $x = -\infty$ to $x = -1$; decreases from $x = -1$ to $x = -\frac{1}{3}$; increases from $x = -\frac{1}{3}$ to $x = \infty$.

2. Find the intervals of increasing and decreasing for the function

$$y = x^3 - 2x^2 + x - 4,$$

and show where the curve is parallel to the x-axis.

3. At how many points can the slope of the tangent to the curve

$$y = 2x^3 - 3x^2 + 1$$

be 1? -1? Find the points.

4. Compute the angle at which the following curves intersect:

$$y = 3x^2 - 1, \quad y = 2x^2 + 3.$$

29. Turning values of a function. It follows that the values of x at which $\phi(x)$ ceases to increase and begins to decrease are those at which $\phi'(x)$ changes sign from positive to negative; and that the values of x at which $\phi(x)$ ceases to decrease and begins to increase are those at which $\phi'(x)$ changes its sign from negative to positive. In the former case, $\phi(x)$ is said to pass through a *maximum*, in the latter, a *minimum*, value.

Ex. **1.** Find the turning values of the function

$$\phi(x) = 2x^3 - 3x^2 - 12x + 4,$$

and exhibit the mode of variation of the function by sketching the curve

$$y = \phi(x).$$

Here $$\phi'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2),$$

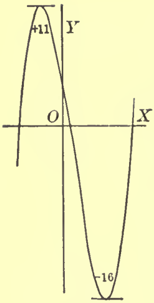

FIG. 9

hence $\phi'(x)$ is negative when x lies between -1 and $+2$, and positive for all other values of x. Thus $\phi(x)$ increases from $x=-\infty$ to $x=-1$; decreases from $x=-1$ to $x=2$; and increases from $x=2$ to $x=\infty$. Hence $\phi(-1)$ is a maximum value of $\phi(x)$, and $\phi(2)$ a minimum.

The general form of the curve $y=\phi(x)$ (Fig. 9) may be inferred from the last statement, and from the following simultaneous values of x and y:

$$x=-\infty,\ -2,\ -1,\ 0,\quad 1,\quad 2,\quad 3,\ 4,\ \infty.$$
$$y=-\infty,\quad 0,\ 11,\ 4,\ -9,\ -16,\ -5,\ 36,\ \infty.$$

Ex. **2.** Exhibit the variation of the function $\phi(x)=(x-1)^{\frac{2}{3}}+2$, especially its turning values.

Since $\phi'(x)=\dfrac{2}{3}\dfrac{1}{(x-1)^{\frac{1}{3}}}$,

hence $\phi'(x)$ changes sign at $x=1$, being negative when $x<1$, infinite if $x=1$, and positive if $x>1$. Thus $\phi(1)=2$ is a minimum turning value of $\phi(x)$. The graph of the function is as shown in Fig. 10, with a vertical tangent at the point (1, 2).

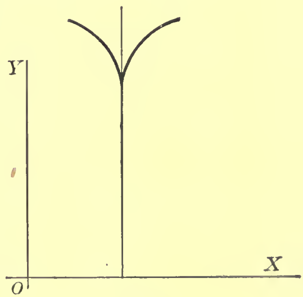

FIG. 10

Ex. **3.** Examine for maxima and minima the function

$$\phi(x)=(x-1)^{\frac{1}{3}}+1.$$

Here $\quad \phi'(x)=\dfrac{1}{3}\dfrac{1}{(x-1)^{\frac{2}{3}}}$,

hence $\phi'(x)$ never changes sign, but is always positive. There is accordingly no turning value. The curve $y=\phi(x)$ has a vertical tangent at the point (1, 1), since $\dfrac{dy}{dx}$ is infinite when $x=1$. (Fig. 11.)

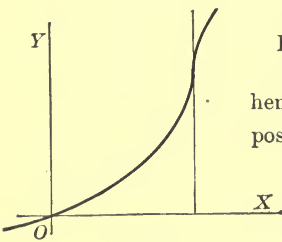

FIG. 11

30. Critical values of the variable. It has been shown that the necessary and sufficient condition for a turning value of $\phi(x)$ is that $\phi'(x)$ shall change its sign. Now a function can change its sign only when it passes through zero, as in Ex. 1 (Art. 29), or when its reciprocal passes through zero, as in Ex. 2. In the latter case it is usual to say that the function passes through infinity. It is not true, conversely, that a function always changes its sign in passing through zero or infinity, *e.g.* x^2 and x^{-2}.

Nevertheless all the values of x, at which $\phi'(x)$ passes through zero or infinity, are called *critical* values of x, because they are to be further examined to determine whether $\phi'(x)$ actually changes sign as x passes through each such value; and whether, in consequence, $\phi(x)$ passes through a turning value.

For instance, in Ex. 1, the derivative $\phi'(x)$ vanishes when $x = -1$, and when $x = 2$, and it does not become infinite for any finite value of x. Thus the critical values are -1, 2, both of which give turning values to $\phi(x)$. Again, in Exs. 2, 3, the critical value is $x = 1$, since it makes $\phi'(x)$ infinite; it gives a turning value to $\phi(x)$ in Ex. 2, but not in Ex. 3.

31. Method of determining whether $\phi'(x)$ changes its sign in passing through zero or infinity. Let a be a critical value of x; in other words, let $\phi'(a)$ be either zero or infinite, and let h be a very small positive number, so that $a - h$ and $a + h$ are two numbers very close to a, and on opposite sides of it. In order to determine whether $\phi'(x)$ changes sign as x increases through the value a, it is necessary only to compare the signs of $\phi'(a + h)$ and $\phi'(a - h)$. If it is possible to take h so small that $\phi'(a-h)$ is positive and $\phi'(a+h)$ negative, then $\phi'(x)$ changes sign as x passes through the value a, and $\phi(x)$ passes through a maximum value $\phi(a)$. Similarly, if $\phi'(a-h)$ is negative and $\phi'(a+h)$ positive, then $\phi(x)$ passes through a minimum value $\phi(a)$.

If $\phi'(a-h)$ and $\phi'(a+h)$ have the same sign, however small h may be, then $\phi(a)$ is not a turning value of $\phi(x)$.

Ex. Find the turning values of the function

$$\phi(x)=(x-1)^2(x+1)^3.$$

Here
$$\begin{aligned}\phi'(x)&=2(x-1)(x+1)^3+3(x-1)^2(x+1)^2\\&=(x-1)(x+1)^2(5x-1).\end{aligned}$$

Hence $\phi'(x)$ becomes zero at $x=-1, \frac{1}{5}$, and 1; it does not become infinite for any finite value of x.

Thus, the critical values are $-1, \frac{1}{5}, 1$.

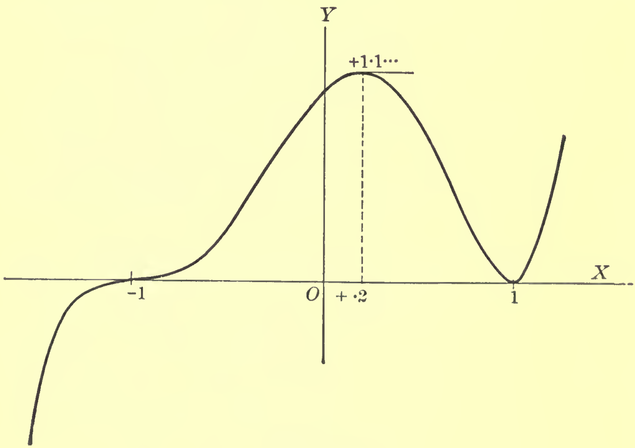

FIG. 12

When x has any value less than -1, the three factors of $\phi'(x)$ take the signs $- + -$, hence $\phi'(x)$ is $+$, and when x has a value between -1 and $\frac{1}{5}$ they become $- + -$, and $\phi'(x)$ is still $+$; hence $\phi(-1) = 0$ is not a turning value of $\phi(x)$.

When x has any value between $\frac{1}{5}$ and 1, the signs are $- + +$ and $\phi'(x)$ is $-$; hence $\phi(\frac{1}{5})$ is a maximum.

Finally, if x has any value greater than 1, the signs are $+ + +$; hence $\phi'(x)$ changes sign from $-$ to $+$ as x increases through 1, and $\phi(1) = 0$ is a minimum value of $\phi(x)$.

The general march of the function may be exhibited graphically by tracing the curve $y = \phi(x)$ (Fig. 12), using the foregoing results and observing the following simultaneous values of x and y:

$$\begin{array}{llllllll} x = -\infty, & -2, & -1, & 0, & \frac{1}{5}, & 1, & 2, & \infty. \\ y = -\infty, & -9, & 0, & 1, & 1 \cdot 1 \cdots, & 0, & 27, & \infty. \end{array}$$

32. Second method of determining whether $\phi'(x)$ changes sign in passing through zero. The following method may be employed when the function and its derivatives are continuous in the vicinity of the critical value $x = a$.

Suppose, when x increases through the value a, that $\phi'(x)$ changes sign from positive through zero to negative. Its change from positive to zero is a decrease, and so is the change from zero to negative; thus $\phi'(x)$ is a decreasing function at $x = a$, and hence its derivative $\phi''(x)$ is negative at $x = a$.

On the other hand, if $\phi'(x)$ changes sign from negative through zero to positive, it is an increasing function and $\phi''(x)$ is positive at $x = a$; hence:

The function $\phi(x)$ has a maximum value $\phi(a)$, when $\phi'(a) = 0$ and $\phi''(a)$ is negative; $\phi(x)$ has a minimum value $\phi(a)$, when $\phi'(a) = 0$ and $\phi''(a)$ is positive.

It may happen, however, that $\phi''(a)$ is also zero.

To determine in this case whether $\phi(x)$ has a turning value, it is necessary to proceed to the higher derivatives. If $\phi(x)$ is a maximum, $\phi''(x)$ is negative just before vanishing, and negative just after, for the reason given above; but the change from negative to zero is an increase, and the change from zero to negative is a decrease; thus $\phi''(x)$ changes from increasing to decreasing as x passes through a. Hence $\phi'''(x)$ changes sign from positive through zero to negative, and it follows, as before, that its derivative $\phi^{\text{IV}}(x)$ is negative.

Thus $\phi(a)$ is a maximum value of $\phi(x)$ if $\phi'(a)=0$, $\phi''(a)=0$, $\phi'''(a)=0$, $\phi^{\text{IV}}(a)$ negative. Similarly, $\phi(a)$ is a minimum value of $\phi(x)$ if $\phi'(a)=0$, $\phi''(a)=0$, $\phi'''(a)=0$, and $\phi^{\text{IV}}(a)$ positive.

If it happens that $\phi^{\text{IV}}(a)=0$, it is necessary to proceed to still higher derivatives to test for turning values. The result may then be generalized as follows:

The function $\phi(x)$ has a maximum (or minimum) value at $x=a$ if one or more of the derivatives $\phi'(a)$, $\phi''(a)$, $\phi'''(a)$ vanish and if the first one that does not vanish is of even order, and negative (or positive).

Ex. Find the critical values in the example of Art. 31 by the second method.

$$\phi''(x) = (x+1)^2(5x-1)+2(x-1)(x+1)(5x-1)+5(x-1)(x+1)^2$$
$$= 4(5x^3+3x^2-3x-1),$$

$\phi''(1)=16$, hence $\phi(1)$ is a minimum value of $\phi(x)$;

$\phi''(-1)=0$, hence it is necessary to find $\phi'''(-1)$;

$\phi'''(x) = 12(5x^2+2x-1)$,

$\phi'''(-1)=24$, hence $\phi(-1)$ is neither a maximum nor a minimum value of $\phi(x)$.

Again, $\phi''(\frac{1}{5}) = 5(\frac{1}{5}-1)(\frac{1}{5}+1)^2$ is negative, hence $\phi(\frac{1}{5})$ is a maximum value of $\phi(x)$.

33. The maxima and minima of any continuous function occur alternately. It has been seen that the maximum and minimum values of a rational polynomial occur alternately when the variable is continually increased, or diminished.

This principle is true also in the case of every continuous function of a single variable. For, let $\phi(a)$, $\phi(b)$ be two maximum values of $\phi(x)$, in which a is supposed less than b. Then, when $x = a + h$, the function is decreasing; when $x = b - h$, the function is increasing, h being taken sufficiently small and positive. But in passing from a decreasing to an increasing state, a continuous function must, at some intermediate value of x, change from decreasing to increasing, that is, must pass through a minimum. Hence, between two maxima there must be at least one minimum.

It can be similarly proved that between two minima there must be at least one maximum.

34. Simplifications that do not alter critical values. The work of finding the critical values of the variable, in the case of any given function, may often be simplified by means of the following self-evident principles.

1. When c is independent of x, any value of x that gives a turning value to $c\phi(x)$, gives a turning value to $\phi(x)$ also; and conversely. These two turning values are of the same or opposite kind according as c is positive or negative.

2. Any value of x that gives a turning value to $c + \phi(x)$ gives a turning value of the same kind to $\phi(x)$ also; and conversely.

3. When n is independent of x, any value of x that gives a turning value to $[\phi(x)]^n$ gives a turning value to $\phi(x)$ also; and conversely. These turning values are of the same or opposite kind according as $n[\phi(x)]^{n-1}$ is positive or negative.

EXERCISES

Find the critical values of x in the following functions, determine the nature of the function at each, and obtain the graph of the function.

1. $u = x(x^2 - 1)$.

2. $u = 2x^3 - 15x^2 + 36x - 4$.

3. $u = (x-1)^3(x-2)^2$.

4. $u = \sin x + \cos x$.

5. $u = \dfrac{(a-x)^3}{a-2x}$.

6. $u = x(x+1)^2 - 5$.

7. $u = 5 + 12x - 3x^2 - 2x^3$.

8. $u = \dfrac{\log x}{x}$.

9. $u = \sin^2 x \cos^3 x$.

10. $u = \dfrac{x^2 - x + 1}{x^2 + x - 1}$.

11. $u = \dfrac{(x+3)(x+1)}{(x-1)(x-2)}$.

12. Show that a quadratic integral function always has one maximum, or one minimum, but never both.

13. Show that a cubic integral function has in general both a maximum and a minimum value, but may have neither.

14. Show that the function $(x-b)^{\frac{5}{3}}$ has neither a maximum nor a minimum value.

35. Geometric problems in maxima and minima. The theory of the turning values of a function has important applications in solving problems concerning geometric maxima or minima, *i.e.* the determination of the largest or the smallest value a magnitude may have while satisfying certain stated geometric conditions.

The first step is to express the magnitude in question algebraically. If the resulting expression contains more than one variable, the stated conditions will furnish enough relations between these variables, so that all the others may be expressed in terms of one. The expression to be maximized or minimized, being thus made a function of a single variable, can be treated by the preceding rules.

Ex. **1**. Find the largest rectangle whose perimeter is 100. Let x, y denote the dimensions of any of the rectangles whose perimeter is 100. The expression to be maximized is the area

$$u = xy, \tag{1}$$

in which the variables x, y are subject to the stated condition

$$2x + 2y = 100,$$

i.e.

$$y = 50 - x; \tag{2}$$

hence the function to be maximized, expressed in terms of the single variable x, is

$$u = \phi(x) = x(50 - x) = 50x - x^2. \tag{3}$$

The critical value of x is found from the equation

$$\phi'(x) = 50 - 2x = 0$$

to be $x = 25$. When x increases through this value, $\phi'(x)$ changes sign from positive to negative, and hence $\phi(x)$ is a maximum when $x = 25$. Equation (2) shows that the corresponding value of y is 25. Hence the maximum rectangle whose perimeter is 100 is the square whose side is 25.

Ex. **2**. If, from a square piece of tin whose side is a, a square be cut out at each corner, find the side of the latter square in order that the remainder may form a box of maximum capacity, with open top.

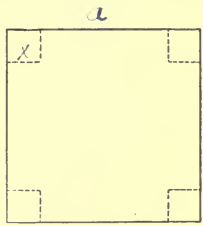

Fig. 13

Let x be a side of each square cut out. Then the bottom of the box will be a square whose side is $a - 2x$, and the depth of the box will be x. Hence the volume is

$$v = x(a - 2x)^2,$$

which is to be made a maximum by varying x.

Here

$$\begin{aligned}\frac{dv}{dx} &= (a - 2x)^2 - 4x(a - 2x)\\ &= (a - 2x)(a - 6x).\end{aligned}$$

This derivative vanishes when $x = \frac{a}{2}$, and when $x = \frac{a}{6}$. It will be found, by applying the usual test, that $x = \frac{a}{2}$ gives v the minimum value zero, and that $x = \frac{a}{6}$ gives it the maximum value $\frac{2\,a^3}{27}$. Hence the side of the square to be cut out is one sixth the side of the given square.

Ex. 3. Find the area of the greatest rectangle that can be inscribed in a given ellipse.

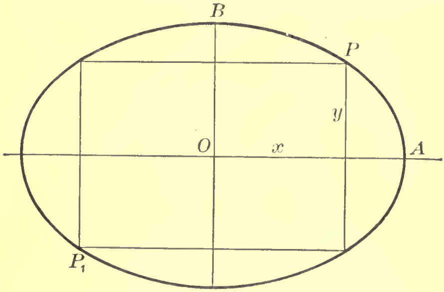

FIG. 14

An inscribed rectangle will evidently be symmetric with regard to the principal axes of the ellipse.

Let a, b denote the lengths of the semi-axes OA, OB (Fig. 14); let $2\,x$, $2\,y$ be the dimensions of an inscribed rectangle. Then the area is

$$u = 4\,xy, \tag{1}$$

in which the variables x, y may be regarded as the coördinates of the vertex P, and are therefore subject to the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2}$$

It is geometrically evident that there is some position of P for which the inscribed rectangle is a maximum.

The elimination of y from (1), by means of (2), gives the function of x to be maximized,

$$u = \frac{4\,b}{a}\,x\sqrt{a^2 - x^2}. \tag{3}$$

By Art. 34, the critical values of x are not altered if this function is divided by the constant $\frac{4\,b}{a}$, and then squared. Hence, the values of x which render u a maximum, give also a maximum value to the function

$$\phi(x) = x^2(a^2 - x^2) = a^2x^2 - x^4.$$

Here

$$\phi'(x) = 2\,a^2x - 4\,x^3 = 2\,x(a^2 - 2\,x^2),$$

$$\phi''(x) = 2\,a^2 - 12\,x^2;$$

hence, by the usual tests, the critical values $x = \pm \dfrac{a}{\sqrt{2}}$ render $\phi(x)$, and therefore the area u, a maximum. The corresponding values of y are given by (2), and the vertex P may be at any of the four points denoted by

$$x = \pm \frac{a}{\sqrt{2}},\ y = \pm \frac{b}{\sqrt{2}},$$

giving in each case the same maximum inscribed rectangle, whose dimensions are $a\sqrt{2}$, $b\sqrt{2}$, and whose area is $2\,ab$, or half that of the circumscribed rectangle.

Ex. **4.** Find the greatest cylinder that can be cut from a given right cone, whose height is h, and the radius of whose base is a.

Let the cone be generated by the revolution of the triangle OAB (Fig. 15), and the inscribed cylinder be generated by the revolution of the rectangle AP.

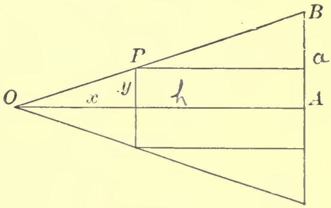

FIG. 15

Let $OA = h$, $AB = a$, and let the coördinates of P be (x, y). Then the function to be maximized is $\pi y^2(h - x)$ subject to the relation $\dfrac{y}{x} = \dfrac{a}{h}$.

This expression becomes

$$V = \frac{\pi a^2}{h^2} \cdot x^2(h - x).$$

The critical value of x is $\frac{2}{3}\,h$, and $V = \dfrac{4\,\pi a^2 h}{27}$.

EXERCISES ON CHAPTER IV

1. What is the width of the rectangle of maximum area that can be inscribed in a given right segment of a parabola?

2. Divide 10 into two parts such that the sum of their squares is a minimum.

3. Find the number that exceeds its square by the greatest possible quantity.

4. What number added to its reciprocal gives the least possible sum?

5. Given the slant height of a right cone; find its altitude when the volume is a maximum.

6. A rectangular piece of pasteboard 30 in. long and 14 in. wide has a square cut out at each corner. Find the side of this square so that the remainder may form a box of maximum contents.

7. Find the altitude of the right cylinder of greatest volume inscribed in a sphere of radius r.

8. Determine the greatest rectangle that can be inscribed in a given triangle whose base is $2b$, and whose altitude is $2a$.

9. A rectangular court is to be built so as to contain a given area c^2, and a wall already constructed is available for one of its sides. Find its dimensions so that the expense incurred in building the walls for the other sides may be the least possible.

10. The volume of a cylinder of revolution being constant, find the relation between its altitude and the radius of its base when the entire surface is a minimum.

11. Assuming that the stiffness of a beam of rectangular cross section varies directly as the breadth and as the cube of the depth, what must be the breadth of the stiffest beam that can be cut from a log 16 in. in diameter?

12. A man who can row 4 mi. per hour, and can walk 5 mi. per hour, is in a boat 3 mi. from the nearest point on a straight beach, and wishes to reach in the shortest time a place on the shore 5 mi. from this point. Where must he land?

13. If the cost per hour for the fuel required to run a given steamer is proportional to the cube of her speed and is \$20 an hour for a speed of 10 knots, and if other expenses amount to \$135 an hour, find the most economical rate at which to run her over a course s.

14. If the cost per hour of running a boat in still water is proportional to the cube of the velocity, find the most economical rate at which to run the steamer upstream against a current of a miles per hour.

15. A Norman window consists of a rectangle surmounted by a semicircle. If the perimeter of the window is given, what must be its proportions in order to admit as much light as possible?

16. Find the most economical proportions for a cylindrical dipper which is to hold a pint.

17. The gate in front of a man's house is 20 yd. from the car track. If the man walks at the rate of 4 mi. an hour and the car on which he is coming home is running at the rate of 12 mi. an hour, where ought he to get off in order to reach home as early as possible?

18. How much water should be poured into a cylindrical tin dipper in order to bring the center of gravity as low down as possible? [Omit until after reading Art. 164.]

19. A statue 10 ft. high stands on a pedestal that is 50 ft. high. How far ought a man whose eyes are 5 ft. above the ground to stand from the pedestal in order that the statue may subtend the greatest possible angle?

20. The sum of the surfaces of a sphere and a cube is given. How do their dimensions compare when the sum of their volumes is a minimum?

21. An electric light is to be placed directly over the center of a circular plot of grass 100 ft. in diameter. Assuming that the intensity of light varies directly as the sine of the angle under which it strikes an illuminated surface and inversely as the square of its distance from the surface, how high should the light be hung in order that the most light possible shall fall on a walk along the circumference of the plot?

22. Find the relation between length of circular arc and radius, in order that the area of a circular sector of a given perimeter shall be a maximum.

23. On the line joining the centers of two mutually external spheres of radii r, R, find the distance of the point from the center of the first sphere from which the maximum of spherical surface is visible.

24. The radius of a circular piece of paper is r. Find the arc of the sector which must be cut from it so that the remaining sector may form the convex surface of a cone of maximum volume.

25. Describe a circle with its center on a given circle so that the length of the arc intercepted within the given circle shall be a maximum.

26. Through a given point within an angle draw a straight line which shall cut off a minimum triangle.

27. What is the length of the axis, and the area, of the maximum parabola which can be cut from a given right circular cone, given that the area of the parabola is equal to two thirds of the product of its base and altitude? A parabola is cut from the cone by a plane parallel to an element.

28. Through the point (a, b) a line is drawn such that the part intercepted between the rectangular coördinate axes is a minimum. Find its length.

29. The lower corner of a leaf, whose edge is a, is folded over so as just to reach the inner edge of the page. Find the width of the part folded over when the length of the crease is a minimum.

30. What is the length of the shortest line that can be drawn tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and having its ends on the coördinate axes?

31. Given a point on the axis of the parabola $y^2 = 2\,px$ at a distance a from the vertex. Find the abscissa of the point of the curve nearest to it.

32. A wall 6 ft. high is parallel to the front of a house and 8 ft. from it. Find the length of the shortest ladder that will reach the house if one end rests on the ground outside the wall.

33. It is required to construct from two circular iron plates of radius a a buoy, composed of two equal cones having a common base, which shall have the greatest possible volume. Find the radius of the base.

34. A weight W is to be raised by means of a lever with force F at one end and the point of support at the other. If the weight is suspended from a point at a distance a from the point of support, and the weight of the beam is w pounds per linear foot, what should be the length of the lever in order that the force required to lift the weight shall be a minimum?

35. A load is hauled up an inclined plane by a horizontal force; it is required to find the inclination θ of the plane so that the mechanical efficiency may be greatest, assuming that the efficiency η is defined by the formula

$$\eta = \frac{\tan \theta}{\tan (\theta + \phi)},$$

where ϕ is the angle of friction; *i.e.* $\tan \phi = \mu$, the coefficient of friction between the load and the plane.

36. If the plane is of cast iron and the load is steel, and if the coefficient of friction between these substances is $\mu = 0.347$, at what angle θ is the efficiency of the inclined plane a maximum?

37. Prove that a conical tent of given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

38. If given currents c and c' produce deflections α and α' in a tangent galvanometer, so that $\tan \alpha / \tan \alpha' = c/c'$, show that $\alpha - \alpha'$ is a maximum when $\alpha + \alpha' = \frac{\pi}{2}$.

CHAPTER V

RATES AND DIFFERENTIALS

36. Rates. Time as independent variable. Suppose a particle P is moving in any path, straight or curved, and let s be the number of space units passed over in t seconds. Then s may be taken as the dependent variable, and t as the independent variable. The motion of P is said to be uniform when equal spaces are passed over in equal times. The number of space units passed over in one second is called the *velocity* of P. The velocity v is thus connected with the space s and the time t by the formula

$$v = \frac{s}{t}.$$

The motion of P is said to be non-uniform when equal spaces are not passed over in equal times. If s is the number of space units passed over in t seconds, then the *average velocity* during these t seconds is defined as $\frac{s}{t}$. If during the time Δt the number of space units Δs are described, then the average velocity during the time Δt is $\frac{\Delta s}{\Delta t}$. The *actual velocity* of P at any instant of time t is the limit which the average velocity approaches as Δt is made to approach zero as a limit.

Thus $$v = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

is the actual velocity of P at the time denoted by t. It is evidently the number of space units that would be passed over in the next second if the velocity remained uniform from the time t to the time $t+1$.

It may be observed that if the more general term, "rate of change," is substituted for the word "velocity," the above statements will apply to any quantity that varies with the time, whether it be length, volume, strength of current, or any other function of the time. For instance, let the quantity of an electric current be C at the time t, and $C+\Delta C$ at the time $t+\Delta t$. Then the *average rate of change* of current in the interval Δt is $\frac{\Delta C}{\Delta t}$; this is the average increase in current-units per second. And the *actual rate of change* at the instant denoted by t is

$$\lim_{\Delta t \doteq 0} \frac{\Delta C}{\Delta t} = \frac{dC}{dt}.$$

This is the number of current-units that would be gained in the next second if the rate of gain were uniform from the time t to the time $t+1$. Since, by Art. 8,

$$\frac{dy}{dx} = \frac{dy}{dt} : \frac{dx}{dt},$$

hence $\frac{dy}{dx}$ measures the ratio of the rates of change of y and of x.

It follows that the result of differentiating

$$y = f(x) \tag{1}$$

may be written in either of the forms

$$\frac{dy}{dx} = f'(x), \tag{2}$$

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}. \tag{3}$$

The latter form is often convenient, and may also be obtained directly from (1) by differentiating both sides with regard to t. It may be read: the rate of change of y is $f'(x)$ times the rate of change of x.

Returning to the illustration of a moving point P, let its coördinates at time t be x and y. Then $\frac{dx}{dt}$ measures the rate of change of the x-coördinate.

Since velocity has been defined as the rate at which a point is moving, the rate $\frac{dx}{dt}$ may be called the velocity which the point P has in the direction of the x-axis, or, more briefly, the x-component of the velocity of P.

It was shown on p. 68 that the actual velocity at any instant t is equal to the space that would be passed over in a unit of time, provided the velocity were uniform during that unit. Accordingly, the x-component of velocity $\frac{dx}{dt}$ may be represented by the distance PA (Fig. 16) which P would pass over in the direction of the x-axis during a unit of time if the velocity remained uniform.

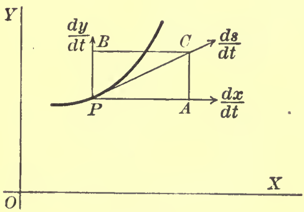

Fig. 16

Similarly $\frac{dy}{dt}$ is the y-component of the velocity of P, and may be represented by the distance PB.

The velocity $\frac{ds}{dt}$ of P along the curve can be represented by the distance PC, measured on the tangent line to the curve at P. It is evident from the parallelogram of velocities that PC is the diagonal of the rectangle PA, PB.

Since $PC^2 = PA^2 + PB^2$, it follows that

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2. \tag{4}$$

Ex. **1.** If a point describes the straight line $3x + 4y = 5$, and if x increases h units per second, find the rates of increase of y and of s.

Since $$y = \tfrac{5}{4} - \tfrac{3}{4}x,$$

hence $$\frac{dy}{dt} = -\frac{3}{4}\frac{dx}{dt}.$$

When $$\frac{dx}{dt} = h,$$

it follows that $\dfrac{dy}{dt} = -\tfrac{3}{4}h$, $\dfrac{ds}{dt} = \sqrt{h^2 + \tfrac{9}{16}h^2} = \tfrac{5}{4}h.$

Ex. **2.** A point describes the parabola $y^2 = 12x$ in such a way that when $x = 3$ the abscissa is increasing at the rate of 2 ft. per second; at what rate is y then increasing? Find also the rate of increase of s.

Since $$y^2 = 12x,$$

then $$2y\frac{dy}{dt} = 12\frac{dx}{dt},$$

$$\frac{dy}{dt} = \frac{6}{y}\frac{dx}{dt} = \frac{6}{\sqrt{12x}}\frac{dx}{dt};$$

hence when $x = 3$ and $\dfrac{dx}{dt} = 2$, it follows that $\dfrac{dy}{dt} = \pm 2$.

Again, $\left(\dfrac{ds}{dt}\right)^2 = \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2$, hence $\dfrac{ds}{dt} = 2\sqrt{2}$ ft. per second.

Ex. **3.** A person is walking toward the foot of a tower on a horizontal plane at the rate of 5 mi. per hour. At what rate is he approaching the top, which is 60 ft. high, when he is 80 ft. from the bottom?

Let x be the distance from the foot of the tower at time t, and y the distance from the top at the same time. Then

$$x^2 + 60^2 = y^2,$$

and

$$x\frac{dx}{dt} = y\frac{dy}{dt}.$$

When x is 80 ft., y is 100 ft.; hence if $\frac{dx}{dt}$ is 5 mi. per hour, $\frac{dy}{dt}$ is 4 mi. per hour.

37. Abbreviated notation for rates. When, as in the above examples, a time derivative is a factor of each member of an equation, it is usually convenient to write, instead of the symbols $\frac{dx}{dt}, \frac{dy}{dt}$, the abbreviations dx and dy, for the rates of change of the variables x and y. Thus the result of differentiating

$$y = f(x) \tag{1}$$

may be written in either of the forms

$$\frac{dy}{dx} = f'(x), \tag{2}$$

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}, \tag{3}$$

$$dy = f'(x)dx. \tag{4}$$

It is to be observed that the last form is not to be regarded as derived from equation (2) by separation of the symbols, dy, dx; for the derivative $\frac{dy}{dx}$ has been defined as the result of performing upon y an indicated operation represented by the symbol $\frac{d}{dx}$, and thus the dy and dx of the symbol $\frac{dy}{dx}$ have been given no separate meaning. The dy and dx of equation (4) stand for the rates, or time derivatives, $\frac{dy}{dt}$ and $\frac{dx}{dt}$ occurring in (3), while the latter equation is itself obtained from (1) by differentiation with regard to t, by Art. 8.

In case the dependence of y upon x is not indicated by a functional operation f, equations (3), (4) take the form

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

$$dy = \frac{dy}{dx}dx.$$

In the abbreviated notation, equation (4) of the last article is written $(ds)^2 = (dx)^2 + (dy)^2$ or $ds^2 = dx^2 + dy^2$.

Ex. **1.** A point describing the parabola $y^2 = 2px$ is moving at the time t with a velocity of v ft. per second. Find the rate of increase of the coördinates x and y at the same instant.

Differentiating the given equation with regard to t, we obtain

$$ydy = pdx.$$

But dx, dy also satisfy the relation

$$dx^2 + dy^2 = v^2;$$

hence, by solving these simultaneous equations, we obtain

$$dx = \frac{y}{\sqrt{y^2 + p^2}}v, \qquad dy = \frac{p}{\sqrt{y^2 + p^2}}v, \text{ in feet per second.}$$

Ex. **2.** A vertical wheel of radius 10 ft. is making 5 revolutions per second about a fixed axis. Find the horizontal and vertical velocities of a point on the circumference situated 30° from the horizontal.

Since $\quad x = 10\cos\theta, \qquad y = 10\sin\theta,$

then $\quad dx = -10\sin\theta d\theta, \qquad dy = 10\cos\theta d\theta.$

But $\quad d\theta = 10\pi = 31.416$ radians per second,

hence $\quad dx = -314.16\sin\theta = -157.08$ ft. per second,

and $\quad dy = 314.16\cos\theta = 272.06$ ft. per second.

Ex. **3.** Trace the changes in the horizontal and vertical velocity in a complete revolution.

38. Differentials often substituted for rates. The symbols dx, dy have been defined above as the rates of change of x and y per second.

Sometimes, however, they may conveniently be allowed to stand for any two numbers, large or small, that are proportional to these rates; the equations, being homogeneous in them, will not be affected. It is usual in such cases to speak of the numbers dx and dy by the more general name of *differentials;* they may then be either the rates themselves, or any two numbers in the same ratio.

This will be especially convenient in problems in which the time variable is not explicitly mentioned.

39. Theorem of mean value. Let $f(x)$ be a continuous function of x which has a derivative. It can then be represented by the ordinates of a curve whose equation is $y = f(x)$.

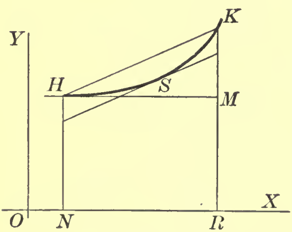

FIG. 17

In Fig. 17, let

$$x = ON, \quad x + h = OR,$$

$$f(x) = NH, \quad f(x+h) = RK.$$

Then $f(x+h) - f(x) = MK$, and

$$\frac{f(x+h) - f(x)}{h} = \frac{MK}{HM} = \tan MHK.$$

But at some point S between H and K the tangent to the curve is parallel to the secant HK. Since the abscissa of S is greater than x and less than $x + h$ it may be represented by $x + \theta h$, in which θ is a positive number less than unity. The slope of the tangent at S is then expressed by $f'(x + \theta h)$, hence

$$\frac{f(x+h) - f(x)}{h} = f'(x + \theta h),$$

from which $$f(x+h) = f(x) + hf'(x + \theta h).$$

The theorem expressed by this formula is known as the *theorem of mean value.*

If in this equation we put

$$f(x+h)-f(x)=dy, \qquad h=dx,$$

in which h is an arbitrary increment, then the relation between the increment of the variable and the actual increment of the function will be expressed by the equation

$$dy=f'(x+\theta dx)\,dx,$$

whereas if dy, dx are regarded as differentials (dy not an actual but a virtual increment), then the relation becomes

$$dy=f'(x)\,dx.$$

This more clearly illustrates that the differential dy is defined as the change that would take place in the function y, corresponding to the actual change dx in the independent variable x, provided the rate of change remained constant.

EXERCISES

1. When x increases from 45° to 45° 15′, find the increase of $\log_{10}\sin x$, assuming that the ratio of the rates of change of the function and the variable remains constant throughout the short interval.

Here $\quad dy = \log_{10} e \cdot \cot x dx = .4343 \cot x dx = .4343\, dx.$

Let $\quad dx = .004363$ (the number of radians in 15′).

Then $\quad dy = .001895,$

which is the approximate increment of $\log_{10}\sin x$.

But $\quad \log_{10}\sin 45° = -\frac{1}{2}\log 2 = -.150515,$

therefore $\quad \log_{10}\sin 45°\,15' = -.148620.$

2. Show that $\log_{10} x$ increases more slowly than x, when $x > \log_{10} e$, that is, $x > 0.4343$.

3. A man is walking at the rate of 5 mi. per hour towards the foot of a tower 60 ft. high standing on a horizontal plane. At what rate is the angle of elevation of the top changing when he is 80 ft. from the foot of the tower?

4. An arc light is hung 12 ft. directly above a straight horizontal walk on which a man 5 ft. in height is walking. How fast is the man's shadow lengthening when he is walking away from the light at the rate of 168 ft. per minute?

5. At what point on the ellipse $16x^2 + 9y^2 = 400$ does y decrease at the same rate that x increases?

6. A vessel is sailing northwest at the rate of 10 mi. per hour. At what rate is she making north latitude?

7. In the parabola $y^2 = 12x$, find the point at which the ordinate and abscissa are increasing equally.

8. At what part of the first quadrant does the angle increase twice as fast as its sine?

9. Find the rate of change in the area of a square when the side b is increasing at a ft. per second.

10. In the function $y = 2x^3 + 6$, what is the value of x at the point where y increases 24 times as fast as x?

11. A circular plate of metal expands by heat so that its diameter increases uniformly at the rate of 2 in. per second. At what rate is the surface increasing when the diameter is 5 in.?

12. What is the value of x at the point at which $x^3 - 5x^2 + 17x$ and $x^3 - 3x$ change at the same rate?

13. Find the points at which the rate of change of the ordinate $y = x^3 - 6x^2 + 3x + 5$ is equal to the rate of change of the slope of the tangent to the curve.

14. The relation between s, the space through which a body falls, and t, the time of falling, is $s = 16\,t^2$. Show that the velocity is equal to $32\,t$.

The rate of change of velocity is called *acceleration* and is denoted by a.

Hence $$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Show that the acceleration of the falling body is a constant.

15. A body moves according to the law $s = \cos(nt + e)$. Show that its acceleration is proportional to the space through which it has moved.

16. If a body is projected upwards in a vacuum with an initial velocity v_0, to what height will it rise, and what will be the time of ascent?

17. A body is projected upwards with a velocity of a ft. per second. After what time will it return?

18. If A is the area of a circle of radius x, show that the circumference is $\frac{dA}{dx}$. Interpret this fact geometrically.

19. A point describing the circle $x^2 + y^2 = 25$ passes through $(3, 4)$ with a velocity of 20 ft. per second. Find its component velocities parallel to the axes.

20. Let a point P move with uniform velocity on a circle of radius a with center O; let AB be any diameter, and Q the orthogonal projection of P on AB. Find an expression for the velocity of Q in terms of the angular velocity of P, and show how this velocity varies during a revolution of P. The motion of the point Q along AB is called harmonic.

21. A point P moves along the curve $y = x^3$ at the rate of 3 ft. per second. At what rate is the angle ϕ, which the tangent to the curve makes with the x-axis, increasing when P is passing through the point $(1, 1)$?

V

CHAPTER VI

DIFFERENTIAL OF AN AREA, ARC, VOLUME, AND SURFACE OF REVOLUTION

40. Differential of an area. If the coördinates of P are (x, y) and those of $Q\ (x+\Delta x, y+\Delta y)$, then $MN = PR = \Delta x$, and $PS = RQ = \Delta y$. If the area $OAPM$ is denoted by A, then A is evidently some function of the abscissa x; also if area $OAQN$ is denoted by $A+\Delta A$ then the area $MNQP$ is ΔA; it is the increment taken by the function A, when x takes the increment Δx. But $MNQP$ lies between the rectangles MR, MQ; hence

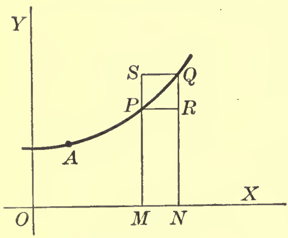

FIG. 18

$$y\Delta x < \Delta A < (y+\Delta y)\Delta x,$$

and
$$y < \frac{\Delta A}{\Delta x} < y + \Delta y.$$

Therefore, when Δx, Δy, ΔA all approach zero,

$$\lim \frac{\Delta A}{\Delta x} = \frac{dA}{dx} = y.$$

Hence, if the ordinate and the area are expressed each as a function of the abscissa, the derivative of the area function with regard to the abscissa is equal to the ordinate function.

In the notation of differentials we may say: *The differential of the area between a curve and the axis of x is measured by the product of the ordinate and the differential of x.*

$$dA = ydx.$$

Ex. If the area included between a curve, the axis of x, and the ordinate whose abscissa is x, is given by the equation

$$A = x^3,$$

find the equation of the curve.

Here
$$y = \frac{dA}{dx} = 3x^2.$$

41. Differential of an arc. A segment of a straight line is measured by applying the unit of measure successively to the segment to be measured. In the case of a curve this is generally impossible. We define the length of a given curve between two points upon it as the limit of the sum of the chords joining points on the curve when the lengths of these chords approach the limit zero. *We shall then assume that the ratio of the arc to the chord approaches the limit* 1 *when the length of the chord approaches the limit zero.* [Compare § 19.]

Let PQ be two points on the curve (Fig. 19); let x, y be the coördinates of P; $x + \Delta x$, $y + \Delta y$ those of Q; s the length of the arc AP; $s + \Delta s$ that of the arc AQ. Draw the ordinates MP, NQ; and draw PR parallel to MN. Then $PR = \Delta x$, $RQ = \Delta y$; arc $PQ = \Delta s$. Hence chord $PQ = \sqrt{(\Delta x)^2 + (\Delta y)^2}$,

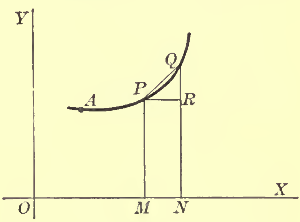

FIG. 19

$$\frac{PQ}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

Therefore
$$\frac{\Delta s}{\Delta x} = \frac{\Delta s}{PQ} \cdot \frac{PQ}{\Delta x} = \frac{\Delta s}{PQ}\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

Taking the limit of both members as Δx approaches zero and putting $\lim\limits_{\Delta x \doteq 0} \dfrac{\Delta s}{PQ} = 1$, we obtain

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{1}$$

Similarly, $$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}. \tag{2}$$

Moreover, from Art. 36,

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2, \tag{3}$$

or in the differential notation,

$$ds^2 = dx^2 + dy^2. \tag{4}$$

42. Trigonometric meaning of $\dfrac{ds}{dx}, \dfrac{ds}{dy}$.

Since $$\frac{\Delta x}{\Delta s} = \frac{\Delta x}{PQ} \cdot \frac{PQ}{\Delta s} = \cos RPQ \cdot \frac{PQ}{\Delta s},$$

it follows by taking the limit that

$$\frac{dx}{ds} = \cos \phi,$$

wherein ϕ, being the limit of the angle RPQ, is the angle which the tangent at the point (x, y) makes with the x-axis.

Similarly, $\dfrac{dy}{ds} = \sin \phi$; whence $\dfrac{ds}{dx} = \sec \phi$, $\dfrac{ds}{dy} = \csc \phi$.

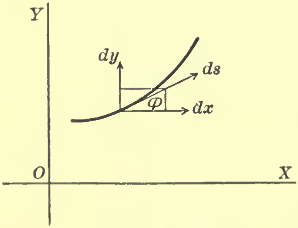

FIG. 20

By using the idea of a rate or differential, all these relations may be conveniently exhibited by Fig. 20.

These results may also be derived from equations (1), (2) of Art. 41, by putting $\dfrac{dy}{dx} = \tan \phi$.

43. Differential of the volume of a solid of revolution. Let the curve APQ (Fig. 21) revolve about the x-axis, and thus generate a surface of revolution; let V be the volume included between this surface, the plane generated by the fixed ordinate at A, and the plane generated by any ordinate MP.

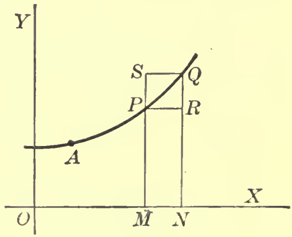

FIG. 21

Let ΔV be the volume generated by the area $PMNQ$. Then ΔV lies between the volumes of the cylinders generated by the rectangles $PMNR$ and $SMNQ$; that is,

$$\pi y^2 \Delta x < \Delta V < \pi (y + \Delta y)^2 \Delta x.$$

Dividing by Δx and taking limits, we obtain

$$\frac{dV}{dx} = \pi y^2, \quad dV = \pi y^2 dx.$$

44. Differential of a surface of revolution. Let S be the area of the surface generated by the arc AP (Fig. 22), and ΔS that generated by the arc PQ, whose length is Δs.

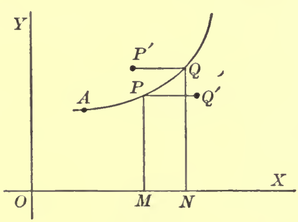

FIG. 22

Draw PQ', QP' parallel to OX and equal in length to the arc PQ. Then it may be assumed as an axiom that the area generated by PQ lies between the areas generated by PQ' and $P'Q$; *i.e.*

$$2\pi y \Delta s < \Delta S < 2\pi (y + \Delta y) \Delta s.$$

Dividing by Δs and passing to the limit,

$$\frac{dS}{ds} = 2\,\pi y, \tag{1}$$

$$\frac{dS}{dx} = \frac{dS}{ds}\cdot\frac{ds}{dx} = 2\,\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}, \tag{2}$$

$$dS = 2\,\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

45. Differential of arc in polar coördinates. Let ρ, θ be the coördinates of P (Fig. 23); $\rho+\Delta\rho$, $\theta+\Delta\theta$ those of Q; s the length of the arc KP; Δs that of the arc PQ; draw PM perpendicular to OQ. Then

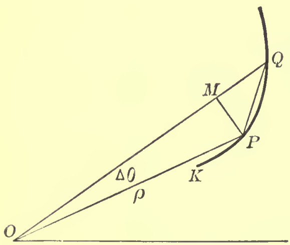

FIG. 23

$$PM = \rho\sin\Delta\theta,$$

$$MQ = OQ - OM = \rho + \Delta\rho - \rho\cos\Delta\theta$$
$$= \rho(1-\cos\Delta\theta) + \Delta\rho$$
$$= 2\,\rho\sin^2\tfrac{1}{2}\,\Delta\theta + \Delta\rho.$$

Hence $\quad PQ^2 = (\rho\sin\Delta\theta)^2 + (2\,\rho\sin^2\tfrac{1}{2}\,\Delta\theta+\Delta\rho)^2,$

$$\left(\frac{PQ}{\Delta\theta}\right)^2 = \rho^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left(\rho\sin\tfrac{1}{2}\,\Delta\theta\cdot\frac{\sin\frac{1}{2}\,\Delta\theta}{\frac{1}{2}\,\Delta\theta}+\frac{\Delta\rho}{\Delta\theta}\right)^2.$$

Replacing the first member by $\left(\dfrac{PQ}{\Delta s}\cdot\dfrac{\Delta s}{\Delta\theta}\right)^2$, passing to the limit when $\Delta\theta \doteq 0$, and putting $\lim\dfrac{PQ}{\Delta s}=1$, $\lim\dfrac{\sin\Delta\theta}{\Delta\theta}=1$, $\lim\dfrac{\sin\frac{1}{2}\,\Delta\theta}{\frac{1}{2}\,\Delta\theta}=1$, we obtain

$$\left(\frac{ds}{d\theta}\right)^2 = \rho^2 + \left(\frac{d\rho}{d\theta}\right)^2;$$

that is, $$\frac{ds}{d\theta} = \sqrt{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2}.$$

In the rate or differential notation this formula may be conveniently written $ds^2 = d\rho^2 + \rho^2 d\theta^2$

46. Differential of area in polar coördinates. Let A be the area of OKP (Fig. 24) measured from a fixed radius vector OK to any other radius vector OP; let ΔA be the area of OPQ. Draw arcs PM, QN, with O as a center. Then the area POQ lies between the areas of the sectors OPM and ONQ; *i.e.*

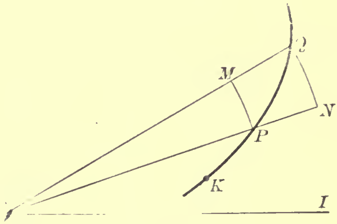

Fig. 24

$$\tfrac{1}{2}\rho^2\Delta\theta < \Delta A < \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta.$$

Dividing by $\Delta\theta$ and passing to the limit, when $\Delta\theta \doteq 0$, we obtain

$$\frac{dA}{d\theta} = \tfrac{1}{2}\rho^2.$$

Hence, in the differential notation we may write the formula

$$dA = \tfrac{1}{2}\rho^2 d\theta.$$

EXERCISES ON CHAPTER VI

1. In the parabola $y^2 = 4ax$, find $\dfrac{ds}{dx}, \dfrac{dA}{dx}, \dfrac{dS}{dx}, \dfrac{dV}{dx}$.

2. Find $\dfrac{ds}{dx}$ and $\dfrac{ds}{dy}$ for the circle $x^2 + y^2 = a^2$.

3. Find $\dfrac{ds}{dx}$ for the curve $e^y \cos x = 1$.

4. Find the x-derivative of the volume of the cone generated by revolving the line $y = ax$ about the axis of x.

5. Find the x-derivative of the volume of the ellipsoid of revolution, formed by revolving $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about its major axis.

6. In the curve $\rho = a^{\theta}$ find $\dfrac{ds}{d\theta}$.

7. Given $\rho = a(1 + \cos\theta)$; find $\dfrac{ds}{d\theta}$.

8. In $\rho^2 \cos 2\theta$, find $\dfrac{ds}{d\theta}$.

9. The parabolic arc $y^2 = 9x$ measured from the vertex to a variable point $P \equiv (x, y)$ is revolving about the x-axis. If P moves along the curve at the rate of 2 in. per second, what is the rate of increase of the surface of revolution when P is passing through the point (4, 6)? What is the rate of increase of the volume of revolution?

10. The radius vector to the cardioid $\rho = 2(1 - \cos\theta)$ is rotating about the origin with an angular velocity of 18° per second. Find the rate at which the extremity P of the radius vector is moving along the curve, taking the inch as unit of length. At what points of the curve will P be moving fastest? slowest? Find the velocities at these points.

CHAPTER VII

APPLICATIONS TO CURVE TRACING

47. Equation of tangent and normal. The function $y=f(x)$ may be represented by a plane curve. It will now be shown how to obtain several of the properties of this curve by means of the principles already established. The tangent line at a point (x_1, y_1) on the curve passes through the point and has the slope $\frac{dy_1}{dx_1}$, the symbol meaning that the coördinates x_1, y_1 are substituted in the first derivative after the differentiation has been performed. Its equation may be written in the form

$$y-y_1=\frac{dy_1}{dx_1}(x-x_1). \tag{1}$$

The *normal* to the curve at the point (x_1, y_1) is the straight line through this point, perpendicular to the tangent. Since the slope of the normal is the negative reciprocal of that of the tangent, its equation may be written in the form

$$x-x_1+\frac{dy_1}{dx_1}(y-y_1)=0. \tag{2}$$

48. Length of tangent, normal, subtangent, subnormal. The segments of the tangent and normal intercepted between the point of tangency and the axis OX are called, respectively, the *tangent length* and the *normal length*, and their projections on OX are called the *subtangent* and the *subnormal*.

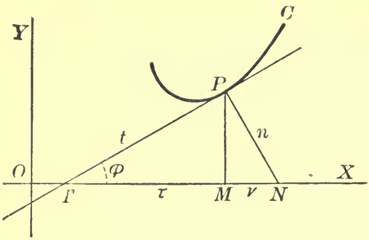

FIG. 25 *a*

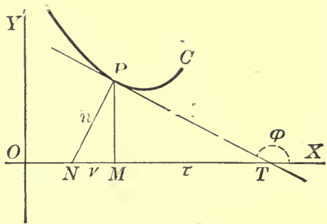

FIG. 25 *b*

Thus, in Fig. 25, *a*, *b* let the tangent and normal to the curve PC at P meet the axis OX in T and N, and let MP be the ordinate of P. Then TP is the tangent length,

PN the normal length,

TM the subtangent,

MN the subnormal.

These will be denoted, respectively, by t, n, τ, ν.

Let the angle XTP be denoted by ϕ, and write $\tan\phi = \dfrac{dy_1}{dx_1}$.

Then $$\frac{dy_1}{dx_1} = \frac{y_1}{\tau} = \frac{\nu}{y_1}, \quad t = \sqrt{y_1^2 + \tau^2}, \quad n = \sqrt{y_1^2 + \nu^2};$$

hence $$\tau = \frac{y_1}{\dfrac{dy_1}{dx_1}}, \quad \nu = y_1 \frac{dy_1}{dx_1}, \quad t = \frac{y\sqrt{1 + \left(\dfrac{dy_1}{dx_1}\right)^2}}{\left(\dfrac{dy_1}{dx_1}\right)},$$

$$n = y_1 \sqrt{1 + \left(\frac{dy_1}{dx_1}\right)^2}.$$

The subtangent is measured from the intersection of the tangent to the foot of the ordinate; it is therefore positive when the foot of the ordinate is to the right of the intersection of tangent. The subnormal is measured from the foot of the ordinate to the intersection of normal, and is positive when the normal cuts OX to the right of the foot of the ordinate. Both are therefore positive or negative, according as ϕ is acute or obtuse.

The expressions for τ, ν may be obtained also by finding from equations (1), (2), Art. 47, the intercepts made by the tangent and normal on the axis OX. The intercept of the tangent subtracted from x_1 gives τ, and x_1 subtracted from the intercept of the normal gives ν.

Ex. Find the intercepts made upon the axes by the tangent at the point (x_1, y_1) on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, and show that their sum is constant.

Differentiating the equation of the curve, we obtain

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0.$$

Hence the equation of the tangent is

$$y - y_1 = -\sqrt{\frac{y_1}{x_1}}(x - x_1).$$

The x intercept is $x_1 + \sqrt{x_1 y_1}$, and the y intercept is $y_1 + \sqrt{x_1 y_1}$, hence their sum is

$$(\sqrt{x_1} + \sqrt{y_1})^2 = a.$$

If a series of lines is drawn such that the sum of the intercepts of each is the same constant, account being taken of the signs, the form of the parabola to which they are all tangent can be readily seen.

EXERCISES

1. Find the equations of the tangent and the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1). Compare the process with that employed in analytic geometry to obtain the same results.

2. Find the equation of the tangent to the curve

$$x^2(x + y) = a^2(x - y)$$

at the origin.

3. Find the equations of the tangent and normal at the point $(1, 3)$ on the curve $y^2 = 9x^3$.

4. Find the equations of the tangent and normal to each of the following curves at the point indicated:

(α) $y = \dfrac{8a^3}{4a^2 + x^2}$, at the point for which $x = 2a$.

(β) $y^2 = 2x^2 - x^3$, at the points for which $x = 1$.

(γ) $y^2 = 4px$, at the point $(p, 2p)$.

5. Find the value of the subtangent of $y^2 = 3x^2 - 12$ at $x = 4$. Compare the process with that given in analytic geometry.

6. Find the length of the tangent to the curve $y^2 = 2x$ at $x = 8$.

7. Find the points at which the tangent is parallel to the axis of x, and at which it is perpendicular to that axis for each of the following curves:

(α) $ax^2 + 2hxy + by^2 = 1$.

(β) $y = \dfrac{x^3 - a^3}{ax}$.

(γ) $y^3 = x^2(2a - x)$.

8. Find the condition that the conics

$$ax^2 + by^2 = 1, \quad a'x^2 + b'y^2 = 1$$

shall cut at right angles.

9. Find the angle at which $x^2 = y^2 + 5$ intersects $8x^2 + 18y^2 = 144$. Compare with Ex. 8.

10. Show that in the equilateral hyperbola $2xy = a^2$ the area of the triangle formed by a variable tangent and the coördinate axes is constant and equal to a^2.

11. At what angle does $y^2 = 8x$ intersect $4x^2 + 2y^2 = 48$?

12. Determine the subnormal to the curve $y^n = a^{n-1}x$.

13. Find the values of x for which the tangent to the curve

$$y^3 = (x - a)^2(x - c)$$

is parallel to the axis of x.

14. Show that the subtangent of the hyperbola $xy = a^2$ is equal to the abscissa of the point of tangency, but opposite in sign.

15. Prove that the parabola $y^2 = 4\,ax$ has a constant subnormal.

16. Show analytically that in the curve $x^2 + y^2 = a^2$ the length of the normal is constant.

17. Show that in the tractrix, the length of the tangent is constant, the equation of the tractrix being

$$x = \sqrt{c^2 - y^2} + \frac{c}{2} \log \frac{c - \sqrt{c^2 - y^2}}{c + \sqrt{c^2 - y^2}}.$$

18. Show that the exponential curve $y = ae^{\frac{x}{c}}$ has a constant subtangent.

19. Find the point on the parabola $y^2 = 4\,px$ at which the angle between the tangent and the line joining the point to the vertex shall be a maximum.

49. Concavity upward and downward. A curve is said to be *concave downward* in the vicinity of a point P when, for a finite distance on each side of P, the curve is situated below

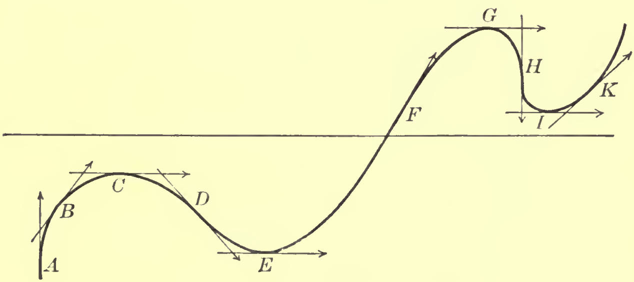

FIG. 26

the tangent drawn at that point, as in the arcs AD, FH. It is *concave upward* when the curve lies above the tangent, as in the arcs DF, HK.

By drawing successive tangents to the curve, as in the figure, we easily see that if the point of contact advances to the right, the tangent swings in the positive direction of rotation when the concavity is upward, and in the negative direction when the concavity is downward. Hence upward concavity may be called a positive bending of the curve, and downward concavity, a negative bending.

A point at which the direction of bending changes continuously from positive to negative, or *vice versa*, as at F or at D, is called a *point of inflexion*, and the tangent at such a point is called a *stationary tangent*.

The points of the curve that are situated just before and just after the point of inflexion are thus on opposite sides of the stationary tangent, and hence the tangent crosses the curve, as at D, F, H.

50. Algebraic test for positive and negative bending. Let the inclination of the tangent line, measured from the positive end of the x-axis toward the forward end of the tangent, be denoted by ϕ. Then ϕ is an increasing or decreasing function of the abscissa according as the bending is positive or negative; for instance, in the arc AD, the angle ϕ diminishes from $+\frac{\pi}{2}$ through zero to $-\frac{\pi}{4}$; in the arc DF, ϕ increases from $-\frac{\pi}{4}$ through zero to $\frac{\pi}{3}$; in the arc FH, ϕ decreases from $+\frac{\pi}{3}$ through zero to $-\frac{\pi}{2}$; and in the arc HK, ϕ increases from $-\frac{\pi}{2}$ through zero to $+\frac{\pi}{4}$.

At a point of inflexion ϕ has evidently a turning value which is a maximum or a minimum, according as the concavity changes from upward to downward, or conversely.

Thus in Fig. 26, ϕ is a maximum at F, and a minimum at D and at H.

Instead of recording the variation of the angle ϕ, it is generally convenient to consider the variation of the slope, $\tan \phi$, which is easily expressed as a function of x by the equation

$$\tan \phi = \frac{dy}{dx}.$$

Since $\tan \phi$ is always an increasing function of ϕ, it follows that the slope function $\frac{dy}{dx}$ is an increasing or a decreasing function of x, according as the concavity is upward or downward, and hence that its x-derivative is positive or negative.

Thus the bending of the curve is in the positive or negative direction of rotation, according as the function $\frac{d^2y}{dx^2}$ is positive or negative.

At a point of inflexion the slope $\frac{dy}{dx}$ is a maximum or a minimum, and therefore its derivative $\frac{d^2y}{dx^2}$ changes sign from positive to negative or from negative to positive. This latter condition is evidently both necessary and sufficient in order that the point (x, y) may be a point of inflexion on the given curve.

Hence, the coördinates of the points of inflexion on the curve

$$y = f(x)$$

may be found by solving the equations

$$f''(x) = 0, \quad f''(x) = \infty,$$

and then testing whether $f''(x)$ changes its sign as x passes through the critical values thus obtained. To any critical value a that satisfies the test corresponds the point of inflexion $(a, f(a))$.

Ex. 1. For the curve $y = (x^2 - 1)^2$

find the points of inflexion, and show the mode of variation of the slope and of the ordinate.

Here
$$\frac{dy}{dx} = 4x(x^2 - 1),$$
$$\frac{d^2y}{dx^2} = 4(3x^2 - 1),$$

hence the critical values for inflexions are $x = \pm\frac{1}{\sqrt{3}}$. It will be seen that as x increases through $-\frac{1}{\sqrt{3}}$, the second derivative changes sign from positive to negative, hence there is an inflexion at which the concavity changes from upward to downward. Similarly, at $x = +\frac{1}{\sqrt{3}}$ the concavity changes from downward to upward. The following numerical table will help to show the mode of variation of the ordinate and of the slope, and the direction of bending.

x	y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
$-\infty$	$+\infty$	$-\infty$	$+$
-2	$+9$	-24	$+$
-1	0	0	$+$
$-\frac{1}{\sqrt{3}}$	$+\frac{4}{9}$	$\frac{8}{3\sqrt{3}}$	0
0	1	0	$-$
$+\frac{1}{\sqrt{3}}$	$+\frac{4}{9}$	$-\frac{8}{3\sqrt{3}}$	0
1	0	0	$+$
$+\infty$	$+\infty$	$+\infty$	$+$

As x increases from $-\infty$ to $-\frac{1}{\sqrt{3}}$ the bending is positive, and the slope continually increases from $-\infty$ through zero to a maximum value $\frac{+8}{3\sqrt{3}}$, which is the slope of the stationary tangent drawn at the point $\left(-\frac{1}{\sqrt{3}}, \frac{4}{9}\right)$.

As x continues to increase from $-\frac{1}{\sqrt{3}}$ to $+\frac{1}{\sqrt{3}}$, the bending is negative, and the slope decreases from $+\frac{8}{3\sqrt{3}}$ through zero to a minimum value $\frac{-8}{3\sqrt{3}}$, which is the slope of the stationary tangent at

$$\left(+\frac{1}{\sqrt{3}}, \frac{4}{9}\right).$$

Finally, as x increases from $+\dfrac{1}{\sqrt{3}}$ to $+\infty$, the bending is positive and the slope increases from the value $-\dfrac{8}{3\sqrt{3}}$ through zero to $+\infty$.

The values $x=-1,\ 0,\ +1$, at which the slope passes through zero, correspond to turning values of the ordinate.

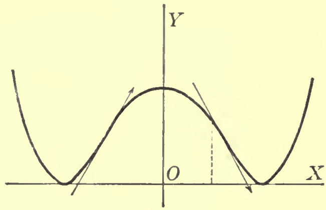

FIG. 27

Ex. **2.** Examine for inflexions the curve

$$x+4=(y-2)^3.$$

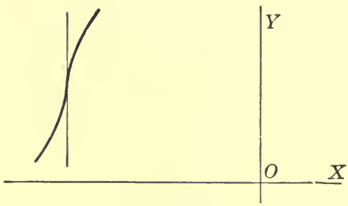

FIG. 28

In this case

$$y=2+(x+4)^{\frac{1}{3}},$$

$$\frac{dy}{dx}=\frac{1}{3}(x+4)^{-\frac{2}{3}},$$

$$\frac{d^2y}{dx^2}=-\frac{2}{9}(x+4)^{-\frac{5}{3}}.$$

Hence, at the point $(-4, 2)$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ are infinite. When $x<-4$, $\dfrac{d^2y}{dx^2}$ is positive, and when $x>-4$, $\dfrac{d^2y}{dx^2}$ is negative.

Thus there is a point of inflexion at $(-4, 2)$, at which the slope is infinite, and the bending changes from the positive to the negative direction.

Ex. **3.** Consider the curve

$$y=x^4.$$

$$\frac{dy}{dx}=4x^3,\quad \frac{d^2y}{dx^2}=12x^2.$$

At $(0, 0)$, $\dfrac{d^2y}{dx^2}$ is zero, but the curve has no inflexion, for $\dfrac{d^2y}{dx^2}$ never changes sign (Fig. 29).

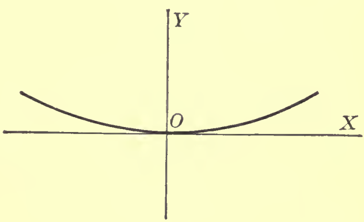

FIG. 29

51. Concavity and convexity toward the axis. A curve is said to be convex or concave toward a line, in the vicinity of a given point on the curve, according as the tangent at the point does or does not lie between the curve and the line, for a finite distance on each side of the point of contact.

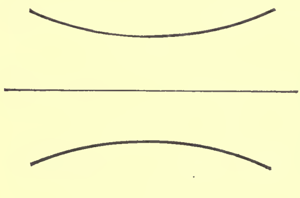

FIG. 30 *a*

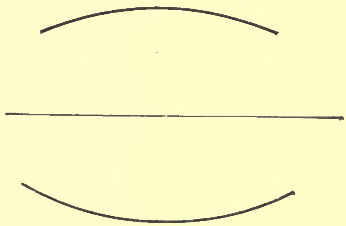

FIG. 30 *b*

First, let the curve be convex toward the x-axis, as in the left-hand figure. Then if y is positive, the bending is positive and $\frac{d^2y}{dx^2}$ is positive; but if y is negative, the bending is negative and $\frac{d^2y}{dx^2}$ is negative. Hence in either case the product $y\frac{d^2y}{dx^2}$ is positive.

Next, let the curve be concave toward the x-axis, as in the right-hand figure. Then if y is positive, the bending is negative and $\frac{d^2y}{dx^2}$ is negative; but if y is negative, the bending is positive and $\frac{d^2y}{dx^2}$ is positive. Thus in either case the product $y\frac{d^2y}{dx^2}$ is negative. Hence:

In the vicinity of a given point (x, y) the curve is convex or concave to the x-axis, according as the product $y\frac{d^2y}{dx^2}$ is positive or negative.

EXERCISES

1. Examine the curve $y = 2 - 3(x - 2)^{\frac{3}{5}}$ for points of inflexion.

2. Show that the curve $a^2y = x(a^2 - x^2)$ has a point of inflexion at the origin.

3. Find the points of inflexion on the curve $y = \dfrac{8\,a^3}{x^2 + 4\,a^2}$.

4. In the curve $ay = x^{\frac{m}{n}}$, prove that the origin is a point of inflexion if m and n are positive odd integers.

5. Show that the curve $y = c \sin \dfrac{x}{a}$ has an infinite number of points of inflexion lying on a straight line.

6. Show that the curve $y(x^2 + a^2) = x$ has three points of inflexion lying on a straight line; find the equation of the line.

7. If $y^2 = f(x)$ is the equation of a curve, prove that the abscissas of its points of inflexion satisfy the equation

$$[f'(x)]^2 = 2f(x) \cdot f''(x).$$

8. Draw the part of the curve $a^2y = \dfrac{x^3}{3} - ax^2 + 2\,a^3$ near its point of inflexion, and find the equation of the stationary tangent.

9. Show that the curve $y = x^{2n}$ has no points of inflexion, n being any positive integer. Sketch the curve.

10. Show that the curve $(1 + x^2)y = 1 - x$ has three points of inflexion, and that they lie in a straight line.

52. Hyperbolic and parabolic branches. When a curve has a branch extending to infinity, the tangents drawn at successive points of this branch may tend to coincide with a definite fixed line, as in the familiar case of the hyperbola. On the other hand, the successive tangents may move farther and farther out of the field, as in the case of the parabola. These two kinds of infinite branches may be called *hyperbolic* and *parabolic*.

The character of each of the infinite branches of a curve can always be determined when the equation of the curve is known.

53. Definition of a rectilinear asymptote. If the tangents at successive points of a curve approach a fixed straight line as a limiting position when the point of contact moves farther and farther along any infinite branch of the given curve, then the fixed line is called an *asymptote* of the curve.

This definition may be stated more briefly but less precisely as follows: An asymptote to a curve is a tangent whose point of contact is at infinity, but which is not itself entirely at infinity.

DETERMINATION OF ASYMPTOTES

54. Method of limiting intercepts. The equation of the tangent at any point (x_1, y_1) being

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1),$$

the intercepts made by this line on the coördinate axes are

$$\left.\begin{aligned} y_0 &= y_1 - x_1 \frac{dy_1}{dx_1}, \\ x_0 &= x_1 - y_1 \frac{dx_1}{dy_1}. \end{aligned}\right\} \tag{1}$$

Suppose the curve has a branch on which $x \doteq \infty$ and $y \doteq \infty$. Then from (1) the limits can be found to which the intercepts x_0, y_0 approach as the coördinates x_1, y_1 of the point of contact tend to become infinite. If these limits are denoted by a, b, the equation of the corresponding asymptote is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Except in special cases this method is usually too complicated to be of practical use in determining the equations of the asymptotes of a given curve. There are two other methods, which together will always suffice to determine the asymptotes of curves whose equations involve only algebraic functions. These may be called the methods of inspection and of substitution.

55. Method of inspection. Infinite ordinates, asymptotes parallel to axes. When an algebraic equation in two coördinates x and y is rationalized, cleared of fractions, and arranged according to powers of one of the coördinates, say y, it takes the form

$$ay^n+(bx+c)y^{n-1}+(dx^2+ex+f)y^{n-2}+\cdots+u_{n-1}y+u_n=0,$$

in which u_n is a polynomial of the degree n in terms of the other coördinate x, and u_{n-1} is of degree $n-1$.

When any value is given to x, the equation determines n values for y.

Let it be required to find for what value of x the corresponding ordinate y has an infinite value.

For this purpose the following theorem from algebra will be recalled:

Given an algebraic equation of degree n,

$$\alpha y^n+\beta y^{n-1}+\gamma y^{n-2}+\cdots=0;$$

if $\alpha=0$, one root y becomes infinite; if $\alpha=0$ and $\beta=0$, two roots y become infinite; and in general if the coefficients of each of the k highest powers of y vanish, the equation will have k infinite roots.

Suppose at first that the term in y^n is present; in other words, that the coefficient a is not zero. Then, when any finite value is given to x, all of the n values of y are finite, and there are accordingly no infinite ordinates for finite values of the abscissa.

Next suppose that a is zero, and b, c, not zero. In this case one value of y is infinite for every finite value of x, and hence the curve passes through the point at infinity on the y axis.

There is one particular value of x, namely, $x = \frac{-c}{b}$, for which an additional root of the equation in y becomes infinite. For, when x has this value, the coefficient $bx + c$ of the highest power of y remaining in the equation vanishes.

Geometrically, every line parallel to the y axis has one point of intersection with the curve at infinity, but the line $bx + c = 0$ has two points of intersection with the curve at infinity. A line having two coincident points of intersection with a curve is a tangent to the curve; and when the coincident points are at infinity, but the line itself not altogether at infinity, the tangent is an asymptote. Hence, an ordinate that becomes infinite for a definite value of x is an asymptote.

Again, if not only a, but also b and c are zero, there are two values of x that make y infinite; namely, those values of x that make $dx^2 + ex + f = 0$. The equations of the infinite ordinates are found by factoring this last equation; and so on.

Similarly, by arranging the equation of the curve according to powers of x, we can easily find what values of y give an infinite value to x.

Ex. **1.** In the curve

$$2x^3 + x^2y + xy^2 = x^2 - y^2 - 5,$$

find the equation of the infinite ordinate, and determine the finite point in which this line meets the curve.

This is a cubic equation in which the coefficient of y^3 is zero.

Arranged in powers of y it is

$$y^2(x + 1) + yx^2 + (2x^3 - x^2 + 5) = 0.$$

When $x = -1$, the equation for y becomes

$$0 \cdot y^2 + y + 2 = 0,$$

the two roots of which are $y = \infty$, $y = -2$; hence the equation of the infinite ordinate is $x + 1 = 0$. The infinite ordinate meets the curve again in the finite point $(-1, -2)$.

Since the term in x^3 is present, there are no infinite values of x for finite values of y.

Ex. **2.** Show that the lines $x = a$, and $y = 0$ are asymptotes to the curve $a^2x = y(x-a)^2$ (Fig. 31).

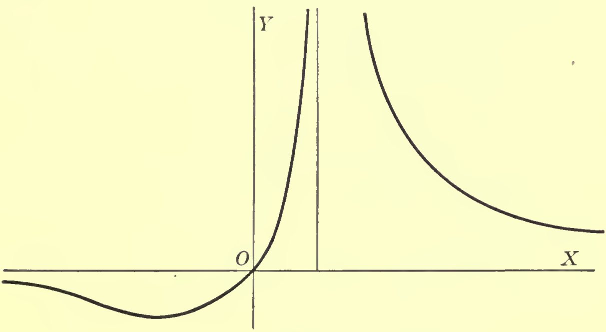

Fig. 31

Ex. **3.** Find the asymptotes of the curve $x^2(y-a) + xy^2 = a^3$.

56. Method of substitution. Oblique asymptotes. The asymptotes that are not parallel to either axis can be found by the method of substitution, which is applicable to all algebraic curves, and is of especial value when the equation is given in the implicit form

$$f(x, y) = 0. \tag{1}$$

Consider the straight line

$$y = mx + b, \tag{2}$$

and let it be required to determine m and b so that this line shall be an asymptote to the curve $f(x, y) = 0$.

Since an asymptote is the limiting position of a line that meets the curve in two points that tend to coincide at infinity, then, by making (1) and (2) simultaneous, the resulting equation in x,

$$f(x, mx + b) = 0,$$

is to have two of its roots infinite. This requires that the coefficients of the two highest powers of x shall vanish. These coefficients, equated to zero, furnish two equations from which the required values of m and b can be determined. These values, substituted in (2), will give the equation of an asymptote.

Ex. **4**. Find the asymptotes to the curve $y^3 = x^2(2a - x)$.

In the first place, there are evidently no asymptotes parallel to either of the coördinate axes. To determine the oblique asymptotes, make the equation of the curve simultaneous with $y = mx + b$, and eliminate y. Then

$$(mx + b)^3 = x^2(2a - x),$$

or, arranged in powers of x,

$$(1 + m^3)x^3 + (3m^2b - 2a)x^2 + 3b^2mx + b^3 = 0.$$

Let $$m^3 + 1 = 0 \text{ and } 3m^2b - 2a = 0.$$

Then $$m = -1,\ b = \frac{2a}{3};$$

hence $$y = -x + \frac{2a}{3}$$

is the equation of an asymptote.

The third intersection of this line with the given curve is found from the equation $3mb^2x + b^3 = 0$, whence $x = \frac{2a}{9}$.

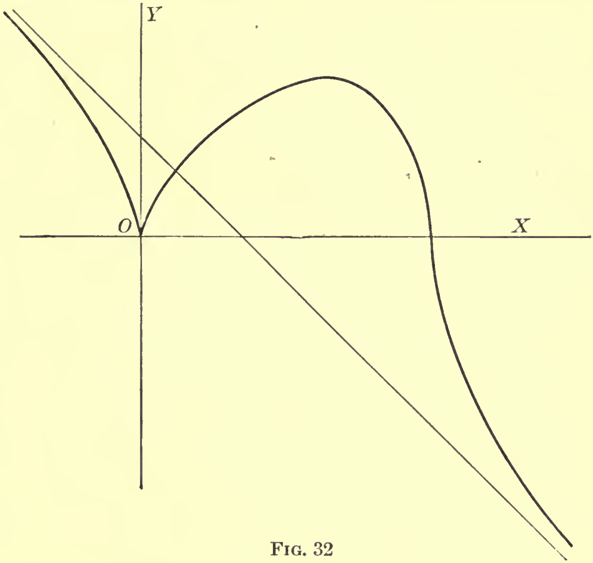

FIG. 32

This is the only oblique asymptote, as the other roots of the equation for m are imaginary.

Ex. **5**. Find the asymptotes to the curve $y(a^2 + x^2) = a^2(a - x)$.

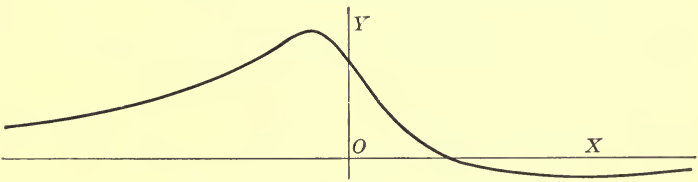

FIG. 33

Here the line $y = 0$ is a horizontal asymptote by Art. 55. To find the oblique asymptotes, put $y = mx + b$.

Then $$(mx + b)(a^2 + x^2) = a^2(a - x),$$

i.e. $$mx^3 + bx^2 + (ma^2 + a^2)x + (a^2b - a^3) = 0;$$

hence $$m = 0,\ b = 0, \quad \text{for an asymptote.}$$

Thus the only asymptote is the line $y = 0$ already found.

57. Number of asymptotes. The illustrations of the last article show that if all the terms are present in the general equation of an nth degree curve, then the equation for determining m is of the nth degree and there are accordingly n values of m, real or imaginary. The equation for finding b is usually of the first degree, but for certain curves one or more values of m may cause the coefficients of x^n and x^{n-1} both to vanish, irrespective of b. In such cases any line whose equation is of the form $y = m_1x + c$ will have two points at infinity on the curve independent of c; but by equating the coefficient of x^{n-2} to zero, two values of b can be found such that the resulting lines have three points at infinity in common with the curve. These two lines are parallel; and it will be seen that in each case in which this happens the equation defining m has a double root, so that the total number of asymptotes is not increased. Hence the total number of asymptotes, real and imaginary, is in general equal to the degree of the equation of the curve.

This number must be reduced whenever a curve has a parabolic branch, since in this case a value of m which makes the coefficient of x^n vanish does not correspond to any finite value of b.

Ex. **6**. Find the asymptotes of the curve $(x - y)^3 = 2x$. The equation in m is $(m - 1)^3 = 0$. The coefficient of x^3 vanishes identically when $m = 1$; that of x is $3(m - 1)b^2 - 2$ which cannot be made to vanish for any finite value of b when $m = 1$. The curve has no asymptotes.

Ex. **7**. Find the asymptotes of the curve

$$y^2 = \frac{(x-1)(2-x)^2}{x-3}$$

and trace the curve. (Fig. 34.)

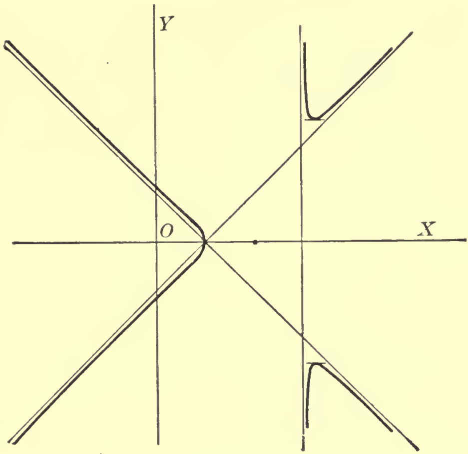

Fig. 34

EXERCISES

Find the asymptotes of each of the following curves:

1. $y(a^2 - x^2) = b(2x + c)$.

2. $y^2 = \dfrac{a^2(x-a)(x-3a)}{x^2 - 2ax}$.

3. $x^2y^2 = a^2(x^2 - y^2)$.

4. $y = a + \dfrac{b^3}{(x-c)^2}$.

5. $y^3 = x^2(a - x)$.

6. $y^2(x-1) = x^2$.

7. $(x+a)y^2 = (y+b)x^2$.

8. $x^2y^2 = x^3 + x + y$.

9. $xy^2 + x^2y = a^3$.

10. $y(x^2 + 3a^2) = x^3$.

11. $x^3 - 3axy + y^3 = 0$.

12. $x^3 + y^3 = a^3$.

13. $x^4 - x^2y^2 + a^2x^2 + b^4 = 0$.

14. $x^4 - y^4 = a^2xy$.

15. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y = 1$.

POLAR COÖRDINATES

58. When the equation of a curve is expressed in polar coördinates, the vectorial angle θ is usually regarded as the independent variable. To determine the direction of the curve at any point, it is most convenient to make use of the angle between the tangent and the radius vector to the point of tangency.

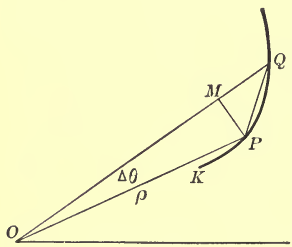

FIG. 35

Let P, Q be two points on the curve (Fig. 35). Join P, Q with the pole O, and drop a perpendicular PM from P on OQ. Let ρ, θ be the coördinates of P; $\rho + \Delta\rho$, $\theta + \Delta\theta$ those of Q. Then the angle $POQ = \Delta\theta$; $PM = \rho \sin \Delta\theta$; and $MQ = OQ - OM = \rho + \Delta\rho - \rho \cos \Delta\theta$.

Hence
$$\tan MQP = \frac{\rho \sin \Delta\theta}{\rho + \Delta\rho - \rho \cos \Delta\theta}.$$

When Q moves to coincidence with P, the angle MQP approaches as a limit the angle between the radius vector and the tangent line at the point P. This angle will be designated by ψ.

Thus
$$\tan \psi = \lim_{\Delta\theta \doteq 0} \frac{\rho \sin \Delta\theta}{\rho + \Delta\rho - \rho \cos \Delta\theta}.$$

But
$$\rho(1 - \cos \Delta\theta) = 2\rho \sin^2 \tfrac{1}{2}\Delta\theta,$$

hence
$$\tan \psi = \lim_{\Delta\theta \doteq 0} \frac{\dfrac{\rho \sin \Delta\theta}{\Delta\theta}}{\rho \sin \frac{1}{2}\Delta\theta \cdot \dfrac{\sin \frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta} + \dfrac{\Delta\rho}{\Delta\theta}}.$$

Since $\lim\limits_{\Delta\theta \doteq 0} \dfrac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta} = 1$, the preceding equation reduces to

$$\tan\psi = \frac{\rho}{\dfrac{d\rho}{d\theta}} = \rho\frac{d\theta}{d\rho}. \tag{3}$$

Ex. **1.** A point describes a circle of radius ρ. Prove that at any instant the arc velocity is ρ times the angle velocity,

i.e., $$\frac{ds}{dt} = \rho\frac{d\theta}{dt}.$$

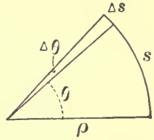

Fig. 36

Ex. **2.** When a point describes a given curve, prove that at any instant the velocity $\dfrac{ds}{dt}$ has a radius component $\dfrac{d\rho}{dt}$ and a component perpendicular to the radius vector $\rho\dfrac{d\theta}{dt}$, and hence that

$$\cos\psi = \frac{d\rho}{ds}, \sin\psi = \rho\frac{d\theta}{ds}, \tan\psi = \rho\frac{d\theta}{d\rho}.$$

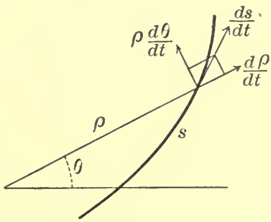

Fig. 37

This furnishes a dynamical proof of equation (3).

59. Relation between $\dfrac{dy}{dx}$ and $\rho\dfrac{d\theta}{d\rho}$. If the initial line is taken as the axis of x, the tangent line at P makes an angle ϕ with this line.

Hence $$\theta + \psi = \phi;$$

i.e., $$\theta + \tan^{-1}\left(\rho\frac{d\theta}{d\rho}\right) = \tan^{-1}\left(\frac{dy}{dx}\right).$$

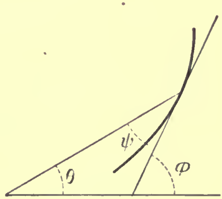

Fig. 38

60. Length of tangent, normal, polar subtangent, and polar subnormal. The portions of the tangent and normal intercepted between the point of tangency P and the line through the pole perpendicular to the radius vector OP, are called the *polar* *tangent length* and the *polar normal length;* their projections on this perpendicular are called the *polar subtangent* and *polar subnormal.*

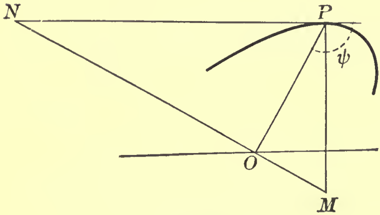

FIG. 39 *a*

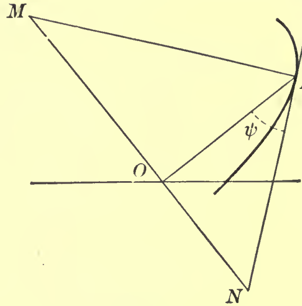

FIG. 39 *b*

Thus, let the tangent and normal at P (Figs. 39 *a*, *b*) meet the perpendicular to OP in the points N and M. Then

PN is the polar tangent length,
PM is the polar normal length,
ON is the polar subtangent,
OM is the polar subnormal.

They are all seen to be independent of the direction of the initial line. The lengths of these lines will now be determined.

Since $PN = OP \cdot \sec OPN = \rho \sec \psi = \rho \sqrt{\rho^2 \left(\frac{d\theta}{d\rho}\right)^2 + 1}$

$$= \rho \frac{d\theta}{d\rho} \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2},$$

hence $\quad$ polar tangent length $= \rho \dfrac{d\theta}{d\rho} \sqrt{\rho^2 + \left(\dfrac{d\rho}{d\theta}\right)^2}.$

Again, $ON = OP \tan OPN = \rho \tan \psi = \rho^2 \dfrac{d\theta}{d\rho},$

hence $\quad$ polar subtangent $= \rho^2 \dfrac{d\theta}{d\rho}.$

$$PM = OP \cdot \csc OPN = \rho \csc \psi = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2},$$

hence polar normal length $= \sqrt{\rho^2 + \left(\dfrac{d\rho}{d\theta}\right)^2}\cdot$

$$OM = OP \cot OPN = \frac{d\rho}{d\theta},$$

hence polar subnormal $= \dfrac{d\rho}{d\theta}\cdot$

The signs of the polar tangent length and polar normal length are ambiguous on account of the radical. The direction of the subtangent is determined by the sign of $\rho^2 \dfrac{d\theta}{d\rho}\cdot$ When $\dfrac{d\theta}{d\rho}$ is positive, the distance ON should be measured to the right, and when negative, to the left of an observer placed at O and looking along OP; for when θ increases with ρ, $\dfrac{d\theta}{d\rho}$ is positive (Art. 28), and ψ is an acute angle (as in Fig. 39 b); when θ decreases as ρ increases, $\dfrac{d\theta}{d\rho}$ is negative, and ψ is obtuse (Fig. 39 a).

EXERCISES

1. In the curve $\rho = a \sin \theta$, find ψ.

2. In the spiral of Archimedes $\rho = a\theta$, show that $\tan \psi = \theta$ and find the polar subtangent, polar normal, and polar subnormal. Trace the curve.

3. Find for the curve $\rho^2 = a^2 \cos 2\theta$ the values of all the expressions treated in this article.

4. Show that in the curve $\rho\theta = a$ the polar subtangent is of constant length. Trace the curve.

5. In the curve $\rho = a(1 - \cos \theta)$, find ψ and the polar subtangent.

6. Show that in the curve $\rho = b \cdot e^{\theta \cot \alpha}$ the tangent makes a constant angle α with the radius vector. For this reason, this curve is called the equiangular spiral.

7. Find the angle of intersection of the curves

$$\rho = a(1 + \cos\theta),\ \rho = b(1 - \cos\theta).$$

8. In the parabola $\rho = a \sec^2\frac{\theta}{2}$, show that $\phi + \psi = \pi$.

EXERCISES ON CHAPTER VII

Trace the following curves. Find asymptotes, intervals of increasing and decreasing ordinate and direction of bending, as well as intercepts on the axes.

1. $y = x^3 + 2x^2 - 7x + 1$.
2. $y^2 = x^3 + 2x^2 - 7x + 1$.
3. $y = (x^2 - 1)^2$.
4. $x^3 + y^3 = 1$.
5. $y^2 = x^3$.
6. $ay^2 = x^3 - bx^2$.
7. $x^4 - y^4 = 2x$.

In the following curves find ψ, determine whether ρ can become infinite, and obtain the (angular) intervals of increasing and decreasing ρ.

8. $\rho = a \cos 2\theta$.
9. $\rho = a \sin 3\theta$.
10. $\rho = a(1 - \cos\theta)$.
11. $\rho = a \sec^2\frac{\theta}{3}$.

CHAPTER VIII

DIFFERENTIATION OF FUNCTIONS OF TWO VARIABLES

Thus far only functions of a single variable have been considered. The present chapter will be devoted to the study of functions of two independent variables x, y. They will be represented by the symbol

$$z = f(x, y).$$

If the simultaneous values of the three variables x, y, z are represented as the rectangular coördinates of a point in space, the locus of all such points is a surface having the equation $z = f(x, y)$.

61. Definition of continuity. A function z of x and y, $z = f(x, y)$, is said to be continuous in the vicinity of any point (a, b) when $f(a, b)$ is real, finite, and determinate, and such that

$$\lim_{\substack{h \doteq 0 \\ k \doteq 0}} f(a + h, b + k) = f(a, b),$$

however h and k approach zero.

When a pair of values a, b exists at which any one of these properties does not hold, the function is said to be discontinuous at the point (a, b).

E.g., let
$$z = \frac{x + y}{x - y}.$$

When $x = 0$, then $z = -1$ for every value of y; when $y = 0$ then $z = +1$ for every value of x. In general, if $y = mx$,

$$z = \frac{1 + m}{1 - m},$$

and z may be made to have any value whatever at (0, 0) by giving an appropriate value to m.

Geometrically speaking, when the point (x, y) moves up to (0, 0), the limiting value of the ordinate z depends upon the direction of approach.

62. Partial differentiation. If in the function

$$z = f(x, y)$$

a fixed value y_1 is given to y, then

$$z = f(x, y_1)$$

is a function of x only, and the rate of change in z caused by a change in x is expressed by

$$dz = \frac{dz}{dx} dx, \tag{1}$$

in which $\frac{dz}{dx}$ is obtained on the supposition that y is constant.

To indicate this fact without the qualifying verbal statement, equation (1) will be written in the form

$$d_x z = \frac{\partial z}{\partial x} dx. \tag{2}$$

The symbol $\frac{\partial z}{\partial x}$ represents the result obtained by differentiating z with regard to x, the variable y being treated as a constant; it is called the *partial derivative* of z with regard to x.

From the definition of differentiation, Art. 6, the partial derivative is the result of the indicated operation

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \doteq 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

Similarly, the symbol $\frac{\partial z}{\partial y}$ represents the result obtained by differentiating z with regard to y, the variable x being treated as a constant; it is called the partial derivative of z with regard to y.

The partial derivative of z with regard to y is accordingly the result of the indicated operation

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \doteq 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

$d_x z = \frac{\partial z}{\partial x} dx$ is called the *partial x-differential* of z, and

$d_y z = \frac{\partial z}{\partial y} dy$ is called the *partial y-differential* of z.

EXERCISES

1. Given $u = x^4 + 3x^2y^2 - 7xy^3$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4u$.

2. Given $u = \tan^{-1}\frac{y}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

3. $u = \log(e^x + e^y)$; find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$.

4. $u = \sin xy$; find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$.

5. $u = \log(x + \sqrt{x^2 + y^2})$; find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

6. $u = \log(\tan x + \tan y + \tan z)$; show that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2.$$

7. $u = \log(x + y)$; show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2}{e^u}$.

8. $u = \frac{xy}{x + y}$; show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$.

9. $u = (y - z)(z - x)(x - y)$; show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

10. $u = \frac{e^{xy}}{e^x + e^y}$; show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = (x + y - 1)u$.

11. $u = \log(x^3 + y^3 + z^3 - 3xyz)$; show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}.$$

63. Total differential. If both x and y are allowed to vary in the function $z = f(x, y)$, the first question that naturally arises is with regard to the meaning of the differential of z.

Let $$z_1 = f(x_1, y_1),$$

and $$z_1 + \Delta z = f(x_1 + \Delta x, y_1 + \Delta y)$$

be two values of the function corresponding to the two pairs of values of the variables x_1, y_1 and $x_1 + \Delta x$, $y_1 + \Delta y$.

The difference

$$\Delta z = f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1, y_1)$$

may be regarded as composed of two parts, the first part being the increment which z takes when x changes from x_1 to $x_1 + \Delta x$, while y remains constant ($y = y_1$), and the second part being the additional increment which z takes when y changes from y_1 to $y_1 + \Delta y$, while x remains constant ($x = x_1 + \Delta x$). The increment Δz may then be written

$$\begin{aligned}\Delta z &= f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1 + \Delta x, y_1) \\ &\qquad\qquad + f(x_1 + \Delta x, y_1) - f(x_1, y_1) \\ &= \frac{f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1 + \Delta x, y_1)}{\Delta y}\Delta y \\ &\qquad\qquad + \frac{f(x_1 + \Delta x, y_1) - f(x_1, y_1)}{\Delta x}\Delta x.\end{aligned}$$

From the theorem of mean value, Art. 39, the last equation may be written

$$\Delta z = \frac{\partial}{\partial x}f(x_1 + \theta\Delta x, y_1)\Delta x + \frac{\partial}{\partial y}f(x_1 + \Delta x, y_1 + \theta_1\Delta y)\Delta y. \quad (3)$$

It represents the actual increment Δz which the dependent variable z takes when the independent variables x and y take the increments Δx and Δy.

To illustrate, let $z = f(x, y)$ be the equation of a surface (Fig. 40). Let $A_1 \equiv (x_1, y_1)$, $A_2 \equiv (x_1 + \Delta x, y_1)$, $A_3 \equiv (x_1 + \Delta x, y_1 + \Delta y)$, so that $A_1P_1 \equiv f(x_1, y_1)$, $A_2P_2 \equiv f(x_1 + \Delta x, y_1)$, $A_3P_3 \equiv f(x_1 + \Delta x, y_1 + \Delta y)$,

$$Q_2P_2 \equiv f(x_1 + \Delta x, y_1) - f(x_1, y_1) = \Delta_1 z,$$
$$Q_3P_3 \equiv f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1 + \Delta x, y_1) = \Delta_2 z,$$
$$R_3P_3 \equiv f(x_1 + \Delta x, y_1 + \Delta y) - f(x_1, y_1) = \Delta_1 z + \Delta_2 z = \Delta z.$$

As the moving point P passes from P to P_2 along the plane curve P_1P_2, the ordinate takes the increment

$$\Delta_1 z = \left(\frac{\partial z}{\partial x}\right)\Delta x,$$

where the derivative is taken at the intermediate point $x = x_1 + \theta\Delta x$, $y = y_1$ (Art. 39). Similarly, as P passes from P_2 to P_3 along the plane curve P_2P_3, the ordinate takes the further increment

$$\Delta_2 z = \left(\frac{\partial z}{\partial y}\right)\Delta y,$$

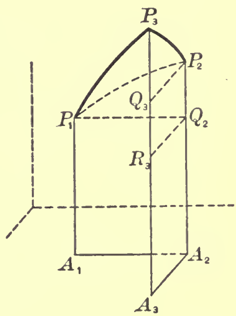

FIG. 40

where the derivative is taken at the intermediate point $y = y_1 + \theta_1\Delta y$, $x = x_1 + \Delta x$.

The sum of these two partial increments gives the total increment Δz.

In the preceding equation (3) let Δx, Δy, Δz be replaced by $\epsilon \cdot dx$, $\epsilon \cdot dy$, $\epsilon \cdot dz$ respectively, in which dx, dy are entirely arbitrary. After removing the common factor ϵ, let ϵ approach zero. The result is

$$dz = \frac{\partial f(x_1, y_1)}{\partial x}\,dx + \frac{\partial f(x_1, y_1)}{\partial y}\,dy. \tag{4}$$

The differential dz defined by this equation is called the *total differential* of z. It is not an actual increment of z, but the increment which z would take if its change continued uniform while x changed from x_1 to $x_1 + dx$ and y changed from y_1 to $y_1 + dy$.

In other words, dz is the rate of change of the variable z when the independent variables x and y change simultaneously at the rates of dx, dy respectively.

Equation (4) may be written in the form

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = d_x z + d_y z,$$

from which the following theorem can be stated; *the total differential of a function of two variables is equal to the sum of its partial differentials taken with regard to the separate variables*, or the total rate of change of z is equal to the sum of its partial rates.

The same method can be applied directly to functions of three or more variables. Thus, if u is a function of the variables x, y, z,

$$u = \phi(x, y, z),$$

then

$$du = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz.$$

Ex. **1**. Given $\qquad z = axy^2 + bx^2y + cx^3 + cy,$

then $\qquad dz = (ay^2 + 2\,bxy + 3\,cx^2)dx + (2\,axy + bx^2 + c)dy.$

Ex. **2**. Given $u = \tan^{-1}\frac{y}{x}$, show that $du = \dfrac{xdy - ydx}{x^2 + y^2}$.

Ex. **3**. Assuming the characteristic equation of a perfect gas, $vp = Rt$, in which v is volume, p pressure, t absolute temperature, and R a constant, express each of the differentials dv, dp, dt, in terms of the other two.

Ex. **4**. A particle moves on the spherical surface $x^2 + y^2 + z^2 = a^2$ in a vertical meridian plane inclined at an angle of 60° to the zx-plane. If the x-component of its velocity is $\frac{a}{10}$ feet per second when $x = \frac{a}{4}$, find the y-component and the z-component velocities.

Since $\qquad z = \sqrt{a^2 - x^2 - y^2},$

then $\qquad dz = -\dfrac{xdx}{\sqrt{a^2 - x^2 - y^2}} - \dfrac{ydy}{\sqrt{a^2 - x^2 - y^2}}.$

But since $dx = \frac{a}{10}$, and the equation of the given meridian plane is $y = x \tan 60°$, hence $dy = \sqrt{3}\,dx = \frac{a\sqrt{3}}{10}$, and $y = \frac{a\sqrt{3}}{4}$. Therefore

$$dz = -\frac{dx}{2\sqrt{3}} - \frac{dy}{2} = -\frac{a\sqrt{3}}{15} \text{ in feet per second.}$$

Ex. **5**. A triangle has a base of 10 units and an altitude of 6 units. The base is made to increase at the rate of 2 units and the altitude to decrease at the rate of $\frac{1}{2}$ unit. At what rate does the area change?

Ex. **6**. A point on the hyperboloid $x^2 - \frac{y^2}{4} - \frac{z^2}{5} = 1$ in the position $x = 2$, $y = 2$ moves so that x increases at the rate of 2 units per second, while y decreases at the rate of 3 units per second. Find the rate of change of z.

Ex. **7**. If the area of a rectangle $A = xy$ is incorrectly measured owing to a small error dx, dy in the length of each side, how close is $dA = xdy + ydx$ to the actual error in the area?

64. Total derivative. If in the relation $z = f(x, y)$, the variables x, y are not independent, but both are functions of another variable s, the process of the preceding article can still be applied. The variable z is now a function of s, and its derivative as to s may be expressed in the form

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}.$$

In particular, if y is not independent, but is a function of x, then s may be chosen as x itself, and the preceding equation becomes

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

If the functional relation between x and y is given,

$$y = \phi(x),$$

then the same result will be obtained, whether $\frac{dz}{dx}$ is determined by the present method, or y is first eliminated from the relation

$$z = f(x, y),$$

and the resulting equation is differentiated as to x. The method of this article frequently shortens the process.

It is here well to note the difference between $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$. The former is the partial derivative of the functional expression for z with regard to x, on the supposition that y is constant. The latter is the total derivative of z with regard to x, when account is taken of the fact that y is itself a function of x.

In the former case the differentiation with regard to x is merely explicit; in the latter it is both explicit and implicit.

Ex. 1. Given $z = \sqrt{x^2 + y^2}$, $y = \log x$; find $\frac{dz}{dx}$.

$$\frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dx},$$

$$\frac{dy}{dx} = \frac{1}{x},$$

hence $$\frac{dz}{dx} = \frac{x^2+y}{x\sqrt{x^2+y^2}}.$$

Ex. 2. If $z = \tan^{-1}\frac{y}{2x}$ and $4x^2 + y^2 = 1$, show that $\frac{dz}{dx} = \frac{-2}{y}$.

65. Differentiation of implicit functions. If, in the relation $z = f(x, y)$, z is assumed to be constant, then

$$dz = 0;$$

hence
$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0, \tag{1}$$

from which
$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \tag{2}$$

In all such cases either variable is an implicit function of the other, and thus the last equation furnishes a rule for finding the derivative of an implicit function.

Ex. **1**. Given $x^3 + y^3 + 3\,axy = c$, find $\frac{dy}{dx}$.

Since $(3\,x^2 + 3\,ay) + (3\,y^2 + 3\,ax)\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}$.

Ex. **2**. $f(ax + by) = c$; $\frac{\partial f}{\partial x} = af'(ax + by)$; $\frac{\partial f}{\partial y} = bf'(ax + by)$;
$$\frac{dy}{dx} = -\frac{a}{b}.$$

Ex. **3**. If $ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0$, find $\frac{dy}{dx}$.

Ex. **4**. Given $x^4 - y^4 = c$, find $\frac{dy}{dx}$.

Ex. **5**. If x increases at the rate of 2 inches per second as it passes through the value $x = 3$ inches, at what rate must y change when $y = 1$ inch in order that the function $2\,xy^2 - 3\,x^2y$ shall remain constant?

If
$$u = 2\,xy^2 - 3\,x^2y,$$

then
$$\frac{\partial u}{\partial x} = 2\,y^2 - 6\,xy, \quad \frac{\partial u}{\partial y} = 4\,xy - 3\,x^2.$$

$$\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = -\frac{2\,y^2 - 6\,xy}{4\,xy - 3\,x^2} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Since $x = 3$, $y = 1$, $\frac{dx}{dt} = 2$, hence $\frac{dy}{dt} = -2\tfrac{2}{15}$ inches per second.

Ex. **6**. $u = v^2 + vy$, $v = \log s$, $y = e^s$. Find $\frac{du}{ds}$.

Ex. **7**. $u = \sin^{-1}(r - s)$, $r = 3t$, $s = 4t^3$. Find $\frac{du}{dt}$.

Ex. **8**. $e^y - e^x + xy = 0$. Find $\frac{dy}{dx}$.

Ex. **9**. $\sin(xy) - e^{xy} - x^2y = 0$. Find $\frac{dy}{dx}$.

It is to be noticed that the result of differentiating any implicit function of x, y by the method of the present article will agree with the result of differentiation according to the rules of Chapter II.

66. Geometric interpretation. Geometrically, the equation $z = f(x, y)$ represents a surface. The equation $y = y_1$ defines a plane parallel to the xz-coördinate plane. The two equations treated simultaneously therefore define the plane section made on the surface $z = f(x, y)$ by the plane $y = y_1$. The derivative $\frac{\partial z_1}{\partial x_1}$ defines the slope of the tangent line to this curve at the point (x_1, y_1, z_1).

Similarly, the plane $x = x_1$ cuts the surface in a section parallel to the yz-coördinate plane. The slope of the tangent line to this second curve is defined by $\frac{\partial z_1}{\partial y_1}$. The equations of these two lines are

$$y = y_1,\ z - z_1 = \frac{\partial z_1}{\partial x_1}(x - x_1),$$

$$x = x_1,\ z - z_1 = \frac{\partial z_1}{\partial y_1}(y - y_1).$$

They have the point (x_1, y_1, z_1) in common; hence the two lines will define a plane. The equation of any plane through the first line will be of the form

$$\left[z - z_1 - \frac{\partial z_1}{\partial x_1}(x - x_1)\right] + \kappa(y - y_1) = 0,$$

and similarly, the equation of any plane through the second line will be of the form

$$\left[z - z_1 - \frac{\partial z_1}{\partial y_1}(y - y_1)\right] + \kappa'(x - x_1) = 0.$$

These two equations will be identical when

$$\kappa = -\frac{\partial z_1}{\partial y_1}, \quad \kappa' = -\frac{\partial z_1}{\partial x_1},$$

hence the equation of the plane containing both lines is

$$z - z_1 = \frac{\partial z_1}{\partial x_1}(x - x_1) + \frac{\partial z_1}{\partial y_1}(y - y_1).$$

It is called the *tangent plane* to the surface $z = f(x, y)$ at the point (x_1, y_1, z_1).

From the equation

$$dz = \frac{\partial z_1}{\partial x_1}dx + \frac{\partial z_1}{\partial y_1}dy, \tag{3}$$

it is seen that if x, y receive the arbitrary increments dx, dy, then the increment dz is defined by the sums of the products of these increments by the corresponding partial derivatives. Thus, if $dx = x - x_1$, $dy = y - y_1$, $dz = z - z_1$, it is seen that the point (x, y, z) always lies in the tangent plane to the surface $z = f(x, y)$, however the increments dx and dy approach zero.

Moreover, the equations of the line joining (x_1, y_1, z_1) to $x_1 + \Delta x$, $y_1 + \Delta y$, $z_1 + \Delta z$ on the surface will be of the form

$$\frac{x - x_1}{\Delta x} = \frac{y - y_1}{\Delta y} = \frac{z - z_1}{\Delta z}.$$

Now as Δx, Δy approach zero, the point always remaining on the surface, the line becomes a tangent in the limit, and its equations are

$$\frac{x - x_1}{dx} = \frac{y - y_1}{dy} = \frac{z - z_1}{dz}, \tag{4}$$

wherein dx, dy depend upon the direction of approach, and dz is defined by (3).

But a tangent line to the surface is also tangent to any plane section passing through the line, and the line (4) is seen to lie in the tangent plane, hence:

The tangent lines to all the plane sections of the surface $z = f(x, y)$ *passing through the point* (x_1, y_1, z_1) *lie in the tangent plane at that point.*

The line through (x_1, y_1, z_1) perpendicular to the tangent plane

$$z - z_1 = \frac{\partial z_1}{\partial x_1}(x - x_1) + \frac{\partial z_1}{\partial y_1}(y - y_1)$$

is called the *normal* to the surface at the point (x_1, y_1, z_1). Its equations are

$$\frac{x - x_1}{\dfrac{\partial z_1}{\partial x_1}} = \frac{y - y_1}{\dfrac{\partial z_1}{\partial y_1}} = \frac{z - z_1}{-1}.$$

If the equation of the surface is given in the implicit form $F(x, y, z) = 0$, then since

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0,$$

the equation of the tangent plane becomes, if $F(x_1, y_1, z_1) = F_1$,

$$\frac{\partial F_1}{\partial x_1}(x - x_1) + \frac{\partial F_1}{\partial y_1}(y - y_1) + \frac{\partial F_1}{\partial z_1}(z - z_1) = 0,$$

and those of the normal are

$$\frac{x-x_1}{\dfrac{\partial F_1}{\partial x_1}}=\frac{y-y_1}{\dfrac{\partial F_1}{\partial y_1}}=\frac{z-z_1}{\dfrac{\partial F_1}{\partial z_1}}.$$

EXERCISES

1. Show that the plane $z=0$ touches the surface $z=xy$ at $(0, 0, 0)$.

2. Find the equation of the tangent plane to the paraboloid $z=2x^2+4y^2$ at the point $(2, 1, 12)$.

3. Find the equations of the normal to the hyperboloid

$$x^2-4y^2+2z^2=6 \text{ at } (2, 2, 3).$$

4. Show that the normal at any point (x_1, y_1, z_1) on the sphere $x^2+y^2+z^2=16$ will pass through the center.

5. Find the equation of the tangent plane at any point (x_1, y_1, z_1) of the surface $x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}=a^{\frac{2}{3}}$ and show that the sum of the squares of the intercepts which it makes on the coördinate axes is constant.

6. Show that the volume of the tetrahedron cut from the cöordinate planes by any tangent plane to the surface $xyz=a^3$ is constant.

7. The sphere $x^2+y^2+z^2=14$ and the ellipsoid $3x^2+2y^2+z^2=20$ pass through the point $(-1, -2, -3)$. Determine the angle at which their tangent planes at this point intersect.

8. How far distant from the origin is the tangent plane to the ellipsoid $x^2+3y^2+2z^2=9$ at the point $(2, -1, 1)$?

9. Find the equation of the tangent plane and of the normal to the cone $z^2=2x^2+y^2$ at (x_1, y_1, z_1) on the surface. Show that the plane will always pass through the vertex of the cone.

10. Find the equations of the tangent line to the circle

$$x^2+y^2+z^2=25,$$
$$x+z=5,$$

at the point $(2, 2\sqrt{3}, 3)$.

67. Successive partial differentiation. The expressions $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ which were defined in Art. 62 are functions of both x and y.

If $\frac{\partial z}{\partial x}$ is differentiated partially as to x, the result is written

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}.$$

This expression is called the second partial derivative of z as to x.

Similarly, the results of the operations indicated by

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right), \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

are written $\frac{\partial^2 z}{\partial y\, \partial x}, \frac{\partial^2 z}{\partial x\, \partial y}, \frac{\partial^2 z}{\partial y^2}$, respectively.

Beginning with the left, we call these expressions the second partial derivative of z as to x and y, the second partial derivative of z as to y and x, and the second partial derivative of z as to y.

68. Order of differentiation indifferent.

Theorem. The successive partial derivatives

$$\frac{\partial^2 z}{\partial y\, \partial x}, \frac{\partial^2 z}{\partial x\, \partial y}$$

are equal for any values of x and y in the vicinity of which z and its first and second partial x- and y-derivatives are continuous.

The truth of this theorem will be assumed. It should be verified for special cases as in the following examples.

Cor. It follows directly that under corresponding conditions the order of differentiation in the higher partial derivatives is indifferent.

E.g.,
$$\frac{\partial^3 z}{\partial x\,\partial y\,\partial x}=\frac{\partial^3 z}{\partial x^2\,\partial y}=\frac{\partial^3 z}{\partial y\,\partial x^2}.$$

EXERCISES

1. Verify that $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial^2 u}{\partial y\,\partial x}$, when $u = x^2y^3$.

2. Verify that $\dfrac{\partial^3 u}{\partial x\,\partial y^2}=\dfrac{\partial^3 u}{\partial y^2\,\partial x}$, when $u = x^2y + xy^3$.

3. Verify that $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial^2 u}{\partial y\,\partial x}$, when $u = y\log(1+xy)$.

4. In Ex. 3 are there any exceptional values of x, y for which the relation is not true?

5. Given $u=(x^2+y^2)^{\frac{1}{2}}$, verify the formula
$$x^2\frac{\partial^2 u}{x^2}+2\,xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

6. Given $u=(x^3+y^3)^{\frac{1}{2}}$, show that the expression in the left member of the differential equation in Ex. 5 is equal to $\dfrac{3\,u}{4}$.

7. Given $u=(x^2+y^2+z^2)^{-\frac{1}{2}}$; prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0.$

8. Given $u=\sec(y+ax)+\tan(y-ax)$; prove that $\dfrac{\partial^2 u}{\partial x^2}=a^2\dfrac{\partial^2 u}{\partial y^2}.$

9. Given $u=\sin x\cos y$; verify that $\dfrac{\partial^4 u}{\partial y^2\,\partial x^2}=\dfrac{\partial^4 u}{\partial x\,\partial y\,\partial x\,\partial y}=\dfrac{\partial^4 u}{\partial x^2\,\partial y^2}.$

10. Given $u=(4\,ab-c^2)^{-\frac{1}{2}}$; prove that $\dfrac{\partial^2 u}{\partial c^2}=\dfrac{\partial^2 u}{\partial a\,\partial b}.$

11. If $u=\dfrac{x^2y^2}{x+y}$, show that $x\dfrac{\partial^2 u}{\partial x^2}+y\dfrac{\partial^2 u}{\partial x\,\partial y}=2\dfrac{\partial u}{\partial x}.$

12. Given $u=\log(x^2+y^2)$, prove $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

13. If $u=(x^n+y^n)^{\frac{1}{n}}$, show that the equation of Ex. 5 is satisfied.

14. Given $u=(x^2+y^2+z^2+w^2)^{-1}$, prove $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}+\dfrac{\partial^2 u}{\partial w^2}=0.$

CHAPTER IX

CHANGE OF VARIABLE

69. Interchange of dependent and independent variables. If y is a continuous function of x, defined by the equation $f(x, y) = 0$, the symbol $\frac{dy}{dx}$ represents the derivative of y with regard to x, when one exists. If x is regarded as a function of y, defined by the same equation, the symbol $\frac{dx}{dy}$ represents the derivative of x with regard to y, when one exists. It is required to find the relation between $\frac{dy}{dx}$ and $\frac{dx}{dy}$.

Let x, y change from the initial values x_1, y_1 to the values $x_1 + \Delta x, y_1 + \Delta y$, subject to the relation $f(x, y) = 0$.

Then, since

$$\frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}},$$

it follows, by taking the limit, that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}. \tag{1}$$

Hence, *if y and x are connected by a functional relation, the derivative of y with regard to x is the reciprocal of the derivative of x with regard to y.*

This process is known as changing the independent variable from x to y. The corresponding relations for the higher derivatives are less simple. They are obtained in the following manner:

To express $\frac{d^2y}{dx^2}$ in terms of $\frac{dx}{dy}, \frac{d^2x}{dy^2}$, differentiate (1) as to x;

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{1}{\frac{dx}{dy}}\right]=\frac{d}{dy}\left[\frac{1}{\frac{dx}{dy}}\right]\cdot\frac{dy}{dx}=\frac{d}{dy}\left[\frac{1}{\frac{dx}{dy}}\right]\cdot\frac{1}{\frac{dx}{dy}}.$$

But
$$\frac{d}{dy}\left[\frac{1}{\frac{dx}{dy}}\right]=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2},$$

hence
$$\frac{d^2y}{dx^2}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}. \tag{2}$$

In a similar manner,
$$\frac{d^3y}{dx^3}=-\frac{\frac{d^3x}{dy^3}\frac{dx}{dy}-3\left(\frac{d^2x}{dy^2}\right)^2}{\left(\frac{dx}{dy}\right)^5}. \tag{3}$$

70. Change of the dependent variable. If y is a function of z, let it be required to express $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots$ in terms of $\frac{dz}{dx}, \frac{d^2z}{dx^2}, \cdots$.

Suppose $y=\phi(z)$. Then

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\phi'(z)\frac{dz}{dx},$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\phi'(z)\frac{dz}{dx}\right)$$

$$=\frac{dz}{dx}\frac{d}{dx}\phi'(z)+\phi'(z)\frac{d^2z}{dx^2}.$$

But $$\frac{d}{dx}\phi'(z)=\frac{d}{dz}\phi'(z)\frac{dz}{dx}=\phi''(z)\frac{dz}{dx}.$$

Hence $$\frac{d^2y}{dx^2}=\phi''(z)\left(\frac{dz}{dx}\right)^2+\phi'(z)\frac{d^2z}{dx^2}. \tag{4}$$

The higher x-derivatives of y can be similarly expressed in terms of x-derivatives of z.

71. Change of the independent variable. Let y be a function of x, and let both x and y be functions of a new variable t. It is required to express $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$, and $\frac{d^2y}{dx^2}$ in terms of $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$.

By Art. 8,
$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \tag{1}$$

hence
$$\frac{d^2y}{dx^2}=\frac{\frac{d^2y}{dt^2}\frac{dx}{dt}-\frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}. \tag{2}$$

In practical examples it is usually better to work by the methods here illustrated than to use the resulting formulas.

72. Simultaneous changes of dependent and of independent variables. Suppose, for example, that an equation involving x, y, $\frac{dy}{dx}$, . . . is given, and it is required to transform the equation into polar coördinates by means of the formulas $x=\rho\cos\theta$, $y=\rho\sin\theta$. Since the variables x and y are connected by some equation (y being a function of x), we may regard x, y, ρ as functions of θ. *E.g.*, consider the function

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

From Art. 71,

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}},$$

and

$$\frac{d^2y}{dx^2}=\frac{\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2}-\frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}}{\left(\frac{dx}{d\theta}\right)^3}.$$

By substituting these values in the expression for R, it becomes

$$R=\frac{\left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2}-\frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}}.$$

This is in terms of a new independent variable θ. We have now to express these θ-derivatives of x and y in terms of ρ and θ.

From the relations $x=\rho\cos\theta$, $y=\rho\sin\theta$ we have

$$\frac{dx}{d\theta}=-\rho\sin\theta+\cos\theta\frac{d\rho}{d\theta},\quad \frac{dy}{d\theta}=\rho\cos\theta+\sin\theta\frac{d\rho}{d\theta},$$

$$\frac{d^2x}{d\theta^2}=-\rho\cos\theta-2\sin\theta\frac{d\rho}{d\theta}+\cos\theta\frac{d^2\rho}{d\theta^2},$$

$$\frac{d^2y}{d\theta^2}=-\rho\sin\theta+2\cos\theta\frac{d\rho}{d\theta}+\sin\theta\frac{d^2\rho}{d\theta^2}.$$

Upon substituting these values in the last expression for R, we obtain

$$R = \frac{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2 + 2\left(\frac{d\rho}{d\theta}\right)^2 - \rho\frac{d^2\rho}{d\theta^2}}.$$

EXERCISES

1. Change the independent variable from x to z in the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0, \text{ when } x = e^z.$$

$$\frac{dy}{dx} = \frac{dy}{dz}\,e^{-z},$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\,e^{-2z} - \frac{dy}{dz}e^{-2z}.$$

Hence $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \text{ becomes } \frac{d^2y}{dz^2} + y = 0.$$

2. Interchange the function and the variable in the equation

$$\frac{d^2y}{dx^2} + 2\,y\left(\frac{dy}{dx}\right)^2 = 0.$$

3. Interchange x and y in the equation

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

4. Change the independent variable from x to y in the equation

$$3\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 = 0.$$

5. Change the dependent variable from y to z in the equation

$$\frac{d^2y}{dx^2} = 1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2, \text{ when } y = \tan z.$$

6. Change the independent variable from x to y in the equation

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+u=0, \text{ when } y=\log x.$$

7. If y is a function of x, and x a function of the time t, express the y-acceleration in terms of the x-acceleration, and the x-velocity.

Since $$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$$

hence $$\frac{d^2y}{dt^2}=\frac{dy}{dx}\frac{d^2x}{dt^2}+\frac{dx}{dt}\cdot\frac{d}{dt}\left(\frac{dy}{dx}\right).$$

But $$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt}=\frac{d^2y}{dx^2}\frac{dx}{dt},$$

hence $$\frac{d^2y}{dt^2}=\frac{dy}{dx}\frac{d^2x}{dt^2}+\frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2.$$

In the abbreviated notation for t-derivatives,

$$d^2y=\frac{dy}{dx}d^2x+\frac{d^2y}{dx^2}(dx)^2.$$

8. Change the independent variable from x to u in the equation

$$\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{y}{(1+x^2)^2}=0, \text{ when } x=\tan u.$$

9. Change the independent variable from x to t in the equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0, \text{ when } x=\cos t.$$

10. Show that the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

remains unchanged in form by the substitution $x=\dfrac{1}{z}$.

11. Interchange the variable and the function in the equation

$$\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right)^3 = 0.$$

12. Change the dependent variable from y to z in the equation

$$\frac{d^2y}{dx^2} + (1 - y)\frac{dy}{dx} + y^2 = 0, \text{ when } y = z^2.$$

Change the independent variable from x to t in the equations:

13. $$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0, \text{ given } x = \cos t.$$

14. $$x^3\frac{d^3v}{dx^3} + 3x^2\frac{d^2v}{dx^2} + x\frac{dv}{dx} + v = 0, \text{ given } x = e^t.$$

15. $$x^4\frac{d^2y}{dx^2} + a^2y = 0, \; x = \frac{1}{t}.$$

16. Transform $\dfrac{x\dfrac{dy}{dx} - y}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}$ by assuming $x = \rho\cos\theta$, $y = \rho\sin\theta$.

17. Given $x = 7 + t^2$, $y = 3 + t^2 - 3t^4$. Find $\dfrac{d^2y}{dx^2}$.

CHAPTER X

EXPANSION OF FUNCTIONS

It is sometimes necessary to expand a given function in a series of powers of the independent variable. For instance, in order to compute and tabulate the successive numerical values of $\sin x$ for different values of x, it is convenient to have $\sin x$ developed in a series of powers of x with coefficients independent of x.

Simple cases of such development have been met with in algebra. For example, by the binomial theorem,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \cdots; \tag{1}$$

and again, by ordinary division,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots. \tag{2}$$

It is to be observed, however, that the series is a proper representative of the function only for values of x within a certain interval. For instance, the identity in (1) holds only for values of x between $-a$ and $+a$ when n is not a positive integer; and the identity in (2) holds only for values of x between -1 and $+1$. In each of these examples, if a finite value outside of the stated limits is given to x, the sum of an infinite number of terms of the series will be infinite, while the function in the first member will be finite.

73. Convergence and divergence of series.* An infinite series is said to be *convergent* or *divergent* according as the sum of the first n terms of the series does or does not approach a finite limit when n is increased without limit.

Those values of x for which a series of powers of x is convergent constitute the *interval of convergence* of the series.

For example, the sum of the first n terms of the geometric series

$$a + ax + ax^2 + ax^3 + \cdots$$

is

$$s_n = \frac{a(1 - x^n)}{1 - x}.$$

First let x be numerically less than unity. Then when n is taken sufficiently large, the term x^n approaches zero;

hence

$$\lim_{n \doteq \infty} s_n = \frac{a}{1 - x}.$$

Next let x be numerically greater than unity. Then x^n becomes infinite when n is infinite; hence, in this case

$$\lim_{n \doteq \infty} s_n = \infty.$$

Thus the given series is convergent or divergent according as x is numerically less or greater than unity. The condition for convergence may then be written

$$-1 < x < 1,$$

and the interval of convergence is between -1 and $+1$.

Similarly the geometric series

$$1 - 3x + 9x^2 - 27x^3 + \cdots,$$

* For an elementary, yet comprehensive and rigorous, treatment of this subject, see Professor Osgood's "Introduction to Infinite Series" (Harvard University Press, 1897).

whose common ratio is $-3x$, is convergent or divergent according as $3x$ is numerically less or greater than unity.

The condition for convergence is $-1 < 3x < 1$, and hence the interval of convergence is between $-\frac{1}{3}$ and $+\frac{1}{3}$.

74. General test for convergence.

Let $$S = u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots$$

be a series of positive terms having the property that $\frac{u_{n+1}}{u_n} < r$ (r a fixed proper fraction) for all values of n that exceed a definite integer k that can be assigned. We wish to prove that under these conditions S is convergent. This is called *the ratio test for convergence.*

According to hypothesis we have the inequalities

$$\frac{u_{k+1}}{u_k} < r, \quad \frac{u_{k+2}}{u_{k+1}} < r, \quad \frac{u_{k+3}}{u_{k+2}} < r, \quad \text{etc.}$$

By multiplying the first two equalities together we obtain $\frac{u_{k+2}}{u_k} < r^2$; then, multiplying this result by the third of the given inequalities we deduce further $\frac{u_{k+3}}{u_k} < r^3$; and so on. These results may be written in the form

$$u_{k+1} < ru_k, \; u_{k+2} < r^2u_k, \; u_{k+3} < r^3u_k, \; \cdots, \; u_{k+p} < r^pu_k.$$

Hence we have the inequality

$$S < u_1 + u_2 + \cdots + u_k + ru_k + r^2u_k + r^3u_k + \cdots$$

But the series in the right member, which may be denoted by S', can be put in the form

$$\begin{aligned} S' &= u_1 + u_2 + \cdots + u_{k-1} + u_k(1 + r + r^2 + r^3 + \cdots) \\ &= u_1 + u_2 + \cdots + u_{k-1} + \frac{u_k}{1-r}. \end{aligned}$$

The terms $u_1, u_2, \cdots, u_k$ being assumed finite, it follows that S' is finite and hence S, which is less than S', also is finite. Since S is formed by the successive addition of positive terms, it follows that the series S converges towards a definite finite limit.

If the series S contains an infinite number of negative, as well as of positive, terms, it converges whenever the series formed by the positive, or *absolute*, values of its terms converges. The series is then said to be *absolutely convergent.*

In order to prove the preceding theorem, we observe that the positive terms of S taken alone form a converging series, whose limit will be denoted by P, and the negative terms taken alone will form a converging series whose limit will be denoted by $-N$. Let S_m denote the sum of the first m terms of S and suppose that these consist of p positive terms whose sum is denoted by P_p and of n negative terms whose sum is $-N_n$. Then we have $S_m = P_p - N_n$. Now when m becomes infinite, p and n also become infinite, and hence

$$S = \lim_{m \doteq \infty} S_m = \lim_{p \doteq \infty} P_p - \lim_{n \doteq \infty} N_n = P - N.$$

Therefore, S is convergent.

When a series is convergent, but the series formed with the absolute values of its terms is not convergent, the given series is said to be *conditionally convergent.**

The *absolute value* of a real number u is its numerical value taken positively, and is written $|u|$.

If a series consists of terms that are alternately positive

* The appropriateness of this terminology is due to the fact that the terms of an absolutely convergent series can be rearranged in any way, without altering the limit of the sum of the series; and that this is not true of a conditionally convergent series. For a simple proof, see Osgood, pp. 43, 44.

and negative, and if, after any definite term of the series, each succeeding term is numerically less than the preceding one, then the series is convergent.

For, suppose that beginning with the term u_k, the series is

$$S' = u_k - u_{k+1} + u_{k+2} - u_{k+3} + u_{k+4} - \cdots,$$

in which u_k, u_{k+1}, etc. represent positive numbers and $u_{k+1} < u_k$, $u_{k+2} < u_{k+1}$, $\cdots$, $u_{m+1} < u_m$, for every value of m greater than k. By grouping the terms in pairs, $(u_k - u_{k+1})$, $(u_{k+2} - u_{k+3})$, $\cdots$, each of which is positive, it is seen that S' has a positive value, which may be finite or infinite.

But S' may also be written in the form

$$S' = u_k - [(u_{k+1} - u_{k+2}) + (u_{k+3} - u_{k+4}) + \cdots],$$

wherein the terms in brackets are all positive, hence S' has a value less than u_k. It therefore converges towards a definite finite limit.

It now follows that the approximate value of S' obtained by algebraically adding u_k, u_{k+1}, $\cdots$, u_m differs from the true value of the series by a number less than u_m. This fact can be shown in precisely the same way as that by which S' has just been shown to have a value less than u_k.

Ex. **1.** Is the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n} + \cdots$ convergent?

Since the terms are alternately positive and negative and their numerical values are always decreasing, it follows at once from the preceding paragraph that this series is convergent. It will be found later that its value is log 2.

Ex. **2.** Prove the convergence of the series met with in Art. 16,

$$2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots.$$

In this case $u_n = \dfrac{1}{n!}$, $u_{n+1} = \dfrac{1}{(n+1)!}$. Hence $\dfrac{u_{n+1}}{u_n} = \dfrac{1}{n+1}$. This ratio is less than $\frac{1}{2}$ for all values of n greater than 2, and the ratio condition for convergence is satisfied.

Ex. **3.** Prove the divergence of the *harmonic* series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

The ratio $u_{n+1} : u_n$ becomes greater than r when n is sufficiently large. By grouping the terms it may be written in the form

$$1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + \cdots,$$

the succeeding groups having $2^3, 2^4, \cdots, 2^n, \cdots$ consecutive terms respectively. The sum of the terms in any group is greater than $\frac{1}{2}$. For, in the nth group the last term $\dfrac{1}{2^n}$ has the least value, and as there are 2^{n-1} terms in the group their sum is greater than $2^{n-1} \cdot \dfrac{1}{2^n} = \dfrac{1}{2}$. As there is an infinity of such groups, their sum is infinite.

Ex. **4.** The series

$$S = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

is convergent for $p > 1$.

Let the terms of S be grouped in the following manner:

$$S = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \cdots,$$

the nth group beginning with $\dfrac{1}{(2^{n-1})^p}$ and containing 2^{n-1} terms. The nth group is accordingly less than its first term multiplied by the number of terms in the group, that is, $< 2^{n-1} \cdot \dfrac{1}{(2^{n-1})^p} = \dfrac{1}{(2^{n-1})^{p-1}}$. Hence we deduce the inequality

$$S < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots,$$

the right member of which is a geometric series having $\dfrac{1}{2^{p-1}}$ as the common ratio. It is therefore convergent, and hence S is convergent, if $\frac{1}{2^{p-1}} < 1$. This inequality is satisfied for every value of p greater than unity. Moreover, it was shown in Ex. 3 that for $p = 1$ the series S is divergent. When $p < 1$, S is divergent. For in that case $\frac{1}{n^p} > \frac{1}{n}$, n is any positive integer (except 1), and therefore the terms of S are greater than the corresponding terms of the harmonic series.

Hence:

The necessary and sufficient condition that the series $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ *may converge is* $p > 1$.

Ex. **5.** Show that the series $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$ is convergent.

This may be proved by comparison with the series in Ex. 4 for the particular case $p = 2$.

Since $$\frac{1}{1 \cdot 2} < 1,\ \frac{1}{2 \cdot 3} < \frac{1}{2^2},\ \frac{1}{3 \cdot 4} < \frac{1}{3^2},\ \cdots,\ \frac{1}{n(n+1)} < \frac{1}{n^2},\ \cdots,$$

it follows that the value of the given series is less than that of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

which is known to be convergent on account of the theorem deduced in the preceding example.

Ex. **6.** Examine for convergence the series whose nth term is $\frac{n}{n^2+1}$.

$$\left[\text{Hint.} \qquad \frac{n}{n^2+1} = \frac{1}{n + \frac{1}{n}} > \frac{1}{n+1}.\right]$$

Ex. **7.** Examine for convergence the series

$$\frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \cdots + \frac{(-1)^{n-1}n}{n^2+1} + \cdots.$$

Ex. **8.** Determine whether the series whose nth term is $\frac{1}{n^2+1}$ is convergent or not; the series whose general term is $\frac{n}{n^3+1}$.

75. Interval of convergence. If the terms $u_1, u_2, \cdots$ of a given series are functions of a variable x, then the series will usually converge for some values of x and diverge for all others. In such a case the problem is to determine the *interval of convergence*, that is, the range of values of x for which the series is convergent. The following examples will illustrate the method of procedure.

Ex. 1. Determine the interval of convergence of the series

$$1 + x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots.$$

In this case $u_n = (n-1)x^{n-1}$ and $u_{n+1} = nx^n$.

Hence,
$$\frac{u_{n+1}}{u_n} = \frac{nx^n}{(n-1)x^{n-1}} = \frac{n}{n-1}x.$$

According to the ratio condition for convergence, it is necessary that this ratio shall be numerically less than 1 for all values of n exceeding a fixed number k. As n increases, the fraction $\frac{n}{n-1}$ approaches unity. Hence if $|x|$ has any fixed value less than 1, the given series is absolutely convergent. The interval of convergence is defined by the inequalities $-1 < x < 1$.

It is evident from the preceding example that the ratio condition for the absolute convergence of a series may be written

$$\lim_{n \doteq \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1, \tag{3}$$

which is especially convenient for application.

Ex. 2. Find the interval of convergence of the series

$$1 + 2 \cdot 2x + 3 \cdot 4x^2 + 4 \cdot 8x^3 + 5 \cdot 16x^4 + \cdots.$$

Here the nth term u_n is $n\,2^{n-1}x^{n-1}$, and the $(n+1)$th term u_{n+1} is $(n+1)\,2^n x^n$;

hence $$\frac{u_{n+1}}{u_n}=\frac{(n+1)2^n x^n}{n\,2^{n-1}x^{n-1}}=\frac{(n+1)\,2\,x}{n},$$

therefore when $n \doteq \infty$, $$\frac{u_{n+1}}{u_n}\doteq 2\,x.$$

It follows by (3) that the series is absolutely convergent when $-1<2\,x<1$, and that the interval of convergence is between $-\frac{1}{2}$ and $+\frac{1}{2}$. The series is evidently not convergent when x has either of the extreme values.

Ex. **3**. Find the interval of convergence of the series

$$\frac{x}{1\cdot 3}-\frac{x^3}{3\cdot 3^3}+\frac{x^5}{5\cdot 3^5}-\frac{x^7}{7\cdot 3^7}+\cdots+\frac{(-1)^{n-1}x^{2n-1}}{(2\,n-1)3^{2n-1}}+\cdots.$$

Here $$\left|\frac{u_{n+1}}{u_n}\right|=\frac{2\,n-1}{2\,n+1}\cdot\frac{3^{2n-1}}{3^{2n+1}}\cdot\frac{x^{2n+1}}{x^{2n-1}}=\frac{2\,n-1}{2\,n+1}\cdot\frac{x^2}{3^2};$$

hence $$\lim_{n\doteq\infty}\left|\frac{u_{n+1}}{u_n}\right|\doteq\frac{x^2}{3^2};$$

thus the series is absolutely convergent when $\frac{x^2}{3^2}<1$, *i.e.*, when $-3<x<3$, and the interval of convergence is from -3 to $+3$. The extreme values of x, in the present case, render the series conditionally convergent.

Ex. **4**. Determine the interval of convergence of the series

$$1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\cdots+(-1)^{n-1}\frac{x^{2n-2}}{(2\,n-2)\,!}+\cdots.$$

Since even powers of x are positive, the terms of this series are alternately positive and negative. The term u_{n+1} is derived from u_n by multiplying it by $\frac{x^2}{(2\,n-1)2\,n}$. For all values of n such that this fraction is less than 1, we shall have the condition $|u_{n+1}|<|u_n|$ and the series is convergent on account of the property of series with alternately positive and negative terms.

Ex. **5.** Prove the convergence of the series

$$x^2 - (x^2 - \tfrac{1}{2}) + (x^2 - \tfrac{1}{2} - \tfrac{1}{4}) - (x^2 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{8}) + \cdots,\ x^2 > 1.$$

In this case $|u_n| = x^2 - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}\right)$. Notice that $\lim\limits_{n \doteq \infty} |u_n|$ is not zero. The series is nevertheless convergent, but not absolutely convergent.

Ex. **6.** Determine the interval of convergence for the series

$$1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n} + \cdots.$$

Ex. **7.** Determine the interval of convergence for the series

$$\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{3}{(x-1)^3} + \cdots + \frac{n}{(x-1)^n} + \cdots.$$

Ex. **8.** Find the interval of convergence for the binomial series

$$1 + ax + \frac{a(a-1)}{1 . 2}x^2 + \frac{a(a-1)(a-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$

in which a is any constant.

Ex. **9.** Show that the series

$$\frac{1}{1^2}\left(\frac{x}{3}\right) - \frac{1}{3^2}\left(\frac{x}{3}\right)^3 + \frac{1}{5^2}\left(\frac{x}{3}\right)^5 - \frac{1}{7^2}\left(\frac{x}{3}\right)^7 + \cdots$$

has the same interval of convergence as that of Ex. 3; but that the extreme values of x render the series absolutely convergent.

76. Remainder after *n* terms. The last article treated of the interval of convergence of a given series without reference to the question whether or not it was the development of any known function. On the other hand, the series that present themselves in this chapter are the developments of given functions, and the first question that arises is concerning those values of x for which the function is equivalent to its development.

When a series has such a generating function, the difference between the value of the function and the sum of the first n terms of its development is called the *remainder after n terms.* Accordingly, if $f(x)$ is the function, $S_n(x)$ the sum of the first n terms of the series, and $R_n(x)$ the remainder obtained by subtracting $S_n(x)$ from $f(x)$, then

$$f(x) = S_n(x) + R_n(x),$$

in which $S_n(x)$, $R_n(x)$ are functions of n as well as of x.

If $$\lim_{n \doteq \infty} R_n(x) = 0, \text{ then } \lim_{n \doteq \infty} S_n(x) = f(x);$$

thus the limit of the series $S_n(x)$ is the generating function when the limit of the remainder is zero. Frequently this is a sufficient test for the convergence of a series.

If a series is expressed in integral powers of $x-a$, the preceding conditions are to be modified by substituting $x-a$ for x; in other respects each criterion is to be applied as before.

77. Maclaurin's expansion of a function in a power-series.* It will now be shown that all the developments of functions in power-series given in algebra and trigonometry are but special cases of one general formula of expansion.

It is proposed to find a formula for the expansion, in ascending positive integral powers of $x-a$, of any assigned function which, with its successive derivatives, is continuous in the vicinity of the value $x = a$.

The preliminary investigation will proceed on the hypothesis that the assigned function $f(x)$ has such a development,

* Named after Colin Maclaurin (1698–1746), who published it in his "Treatise on Fluxions" (1742); but he distinctly says it was known by James Stirling (1692–1770), who also published it in his "Methodus Differentialis" (1730), and by Taylor (see Art. 78).

and that the latter can be treated as identical with the former for all values of x within a certain interval of equivalence that includes the value $x=a$. From this hypothesis the coefficients of the different powers of $x-a$ will be determined. It will then remain to test the validity of the result by finding the conditions that must be fulfilled, in order that the series so obtained may be a proper representation of the generating function.

Let the assumed identity be

$$f(x) \equiv A + B(x-a) + C(x-a)^2 + D(x-a)^3 + E(x-a)^4 + \cdots, \tag{1}$$

in which A, B, C, $\cdots$ are undetermined coefficients, independent of x.

Successive differentiation with regard to x supplies the following additional identities, on the hypothesis that the derivative of each series can be obtained by differentiating it term by term, and that it has some interval of equivalence with its corresponding function:

$$\begin{aligned}
f'(x) &= B + 2C(x-a) + 3D(x-a)^2 + 4E(x-a)^3 + \cdots \\
f''(x) &= 2C + 3\cdot 2D(x-a) + 4\cdot 3E(x-a)^2 + \cdots \\
f'''(x) &= 3\cdot 2D + 4\cdot 3\cdot 2E(x-a) + \cdots \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

If, now, the special value a is given to x, the following equations will be obtained:

$$f(a) = A,\ f'(a) = B,\ f''(a) = 2C,\ f'''(a) = 3\cdot 2D,\ \cdots.$$

Hence,

$$A = f(a),\ B = f'(a),\ C = \frac{f''(a)}{2!},\ D = \frac{f'''(a)}{3!},\ \cdots.$$

The coefficients in (1) are now determined, and the required development is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots. \qquad (2)$$

This is known as *Maclaurin's* series, and the theorem expressed in the formula is called Maclaurin's theorem.

Ex. **1.** Expand $\log x$ in powers of $x - a$, a being positive.

Here $\quad f(x) = \log x, f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{1 \cdot 2}{x^3} \cdots,$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Hence, $\quad f(a) = \log a, f'(a) = \frac{1}{a}, f''(a) = -\frac{1}{a^2}, f'''(a) = \frac{1 \cdot 2}{a^3} \cdots,$

$$f^{(n)}(a) = \frac{(-1)^{n-1}(n-1)!}{a^n}.$$

and, by (2), the required development is

$$\log x = \log a + \frac{1}{a}(x-a) - \frac{1}{2a^2}(x-a)^2 + \frac{1}{3a^3}(x-a)^3 - \cdots + \frac{(-1)^{n-1}}{na^n}(x-a)^n + \cdots.$$

The condition for the convergence of this series is

$$\lim_{n \doteq \infty} \left| \frac{(x-a)^{n+1}}{(n+1)a^{n+1}} : \frac{(x-a)^n}{na^n} \right| < 1,$$

i.e.,
$$\left| \frac{x-a}{a} \right| < 1,$$

$$|x - a| < a,$$

$$0 < x < 2a.$$

This series may be called the *development of log x in the vicinity of* $x = a$. Its development in the vicinity of $x = 1$ has the simpler form

$$\log x = x - 1 - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \cdots,$$

which holds for values of x between 0 and 2.

In using this series for the computation of a table of logarithms we may put for a any number whose logarithm is already known, and for x any number near a in magnitude. It is a great advantage to keep $x - a$ so small that the power-series in $x - a$ may be not merely convergent, but may converge to its limit so rapidly that all powers of $x - a$ above the fourth or fifth may be neglected without affecting the desired degree of accuracy.

E.g., being given $\log 10 = 2.302585$, suppose it is required to compute $\log 11, \log 12, \cdots, \log 20$. Put $a = 10$, and $x = 11$. Then

$$\log 11 = \log 10 + \tfrac{1}{10} - \tfrac{1}{2}(\tfrac{1}{10})^2 + \tfrac{1}{3}(\tfrac{1}{10})^3 - \tfrac{1}{4}(\tfrac{1}{10})^4 + \tfrac{1}{5}(\tfrac{1}{10})^5 - \tfrac{1}{6}(\tfrac{1}{10})^6 + \tfrac{1}{7}(\tfrac{1}{10})^7 \cdots$$

The numerical work may be tabulated in the following form:

$$\begin{array}{rcr}
+\,2.30258509 & \qquad & \\
.10000000 & & -\,.00500000 \\
.00033333 & & .00002500 \\
.00000200 & & .00000017 \\
.00000001 & & \overline{-\,.00502517} \\
\overline{2.40292043} & & \\
-\,.00502517 & & \\
\overline{2.39789526} & &
\end{array}$$

Hence $$\log 11 = 2.397895 \cdots,$$

correct to six places of decimals. To make sure of the sixth figure it is well to carry the work to seven or eight figures. The remaining terms of the series after $\frac{1}{7}(\frac{1}{10})^7$ cannot affect this result, because their sum is less than an infinite decreasing geometric progression whose first term is $\frac{1}{8}(\frac{1}{10})^8$ and whose ratio is $\frac{1}{10}$. From the formula

$$s = \frac{a}{1 - r}$$

it follows that the remainder is less than $\dfrac{1}{72 \cdot 10^9}$.

To calculate $\log 12$, $\log 13$, $\cdots$ we could now keep $a = 10$ and let $x = 12, 13, \cdots$ successively, but in order to secure rapid convergence it is better to change the value of a, choosing for a the nearest number whose logarithm has been found. Thus, in computing log 12 we can use either of the two series

$$\log 12 = \log 10 + \tfrac{2}{10} - \tfrac{1}{2}(\tfrac{2}{10})^2 + \tfrac{1}{3}(\tfrac{2}{10})^3 - \cdots,$$

$$\log 12 = \log 11 + \tfrac{1}{11} - \tfrac{1}{2}(\tfrac{1}{11})^2 + \tfrac{1}{3}(\tfrac{1}{11})^3 - \cdots;$$

but it will be found that five terms of the second furnish as close an approximation as nine terms of the first. The practical advantage of the step-by-step process will depend on how many of the intermediate values we actually require. If we are given log 10 and wish to compute log 15, it may be easier to compute the latter directly without determining the intermediate values.

Ex. **2.** Develop $f(x) = x^3 - 2x^2 + 5x - 7$ in powers of $x - 1$ and use the result to compute $f(1.02), f(1.01), f(.99), f(.98)$.

Ex. **3.** Develop $3y^2 - 14y + 7$ in powers of $y - 2$.

Ex. **4.** Expand $\sin x$ in powers of $x - a$ and use the result to compute sin 31°.

Let $a = 30°$, $x = 31°$. In the formula

$$\sin x = \sin a + \cos a(x - a) - \frac{\sin a}{2!}(x - a)^2 - \frac{\cos a}{3!}(x - a)^3 \cdots,$$

the difference $x - a$ becomes 1° or .001745 radians, and the coefficients of its various powers are all known; since $\sin a = .5$, $\cos a = .866025$ the work is now reduced to numerical calculation in which three terms are sufficient to obtain the result correct to six places of decimals. In general, to calculate $\sin x$ or $\cos x$, take for a the nearest value for which $\sin a$, $\cos a$ are known.

The expansion of a function $f(x)$ in a series of ascending powers of x can be obtained at once from formula (2) by giving a the particular value zero. The series then becomes

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)x^n}{n!} + \cdots. \qquad (3)$$

Ex. **5**. Expand $\sin x$ in powers of x, and find the interval of convergence of the series.

Here
$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0, \\ f'(x) &= \cos x, & f'(0) &= 1, \\ f''(x) &= -\sin x, & f''(0) &= 0, \\ f'''(x) &= -\cos x, & f'''(0) &= -1 \\ f^{\text{IV}}(x) &= \sin x, & f^{\text{IV}}(0) &= 0, \\ f^{\text{V}}(x) &= \cos x, & f^{\text{V}}(0) &= 1, \\ &\cdots\cdots & &\cdots\cdots \end{aligned}$$

Hence, by (3),
$$\sin x = 0 + 1 \cdot x + 0 \cdot x^2 - \frac{1}{3!}x^3 + 0 \cdot x^4 + \frac{1}{5!}x^5 \cdots;$$

thus the required development is

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1} + \cdots.$$

To find the interval of convergence of the series, use the method of Art. 74. The ratio of u_{n+1} to u_n is

$$\frac{u_{n+1}}{u_n} = \frac{x^{2n+1}}{(2n+1)!} : \frac{x^{2n-1}}{(2n-1)!} = \frac{x^2}{(2n+1)2n}.$$

This ratio approaches the limit zero, when n becomes infinite, however large may be the fixed value assigned to x. This limit being less than unity, the series is convergent for any finite value of x, and hence the interval of convergence is from $-\infty$ to $+\infty$.

The preceding series may be used to compute the numerical value of $\sin x$ for any given value of x. It is rapidly convergent when x is small. Take, for example, $x = .5$ radians. Then

$$\sin(.5) = .5 - \frac{(.5)^3}{2 \cdot 3} + \frac{(.5)^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{(.5)^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \cdots,$$

$$\begin{aligned} &= .5000000 \\ &- .0208333 \\ &+ .0002604 \\ &- .0000015 \\ &+ .0000000 \\ \hline \sin(.5) &= .4794256 \cdots \end{aligned}$$

Show that the ratio of u_5 to u_4 is $\frac{1}{288}$; and hence that the error in stopping at u_4 is numerically less than $u_4\left[\frac{1}{288} + \left(\frac{1}{288}\right)^2 + \cdots\right]$, that is, $< \frac{1}{287}u_4$.

When x is not small, it is better to use the more general series in powers of $x - a$.

Ex. **6.** Show that the development of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!} + \cdots,$$

and that the interval of convergence is from $-\infty$ to $+\infty$.

Ex. **7.** Develop the exponential functions a^x, e^x.

Here

$$f(x) = a^x,\ f'(x) = a^x \log a,\ f''(x) = a^x(\log a)^2, \cdots,\ f^{(n)}(x) = a^x(\log a)^n;$$

hence $\quad f(0) = 1;\ f'(0) = \log a,\ f''(0) = (\log a)^2, \cdots, f^{(n)}(0) = (\log a)^n,$

and $$a^x = 1 + (\log a)x + \frac{(\log a)^2}{2!}x^2 + \cdots + \frac{(\log a)^n}{n!}x^n + \cdots.$$

As a special case, put $a = e$.

Then $$\log a = \log e = 1,$$

and $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

These series are convergent for every finite value of x.

Ex. **8.** Compute 10^x when $x = .1$.

Ex. **9.** Compute 10^x when $x = 2.01$.

Ex. **10**. Defining the hyperbolic cosine and the hyperbolic sine by the equations

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \quad \sinh x = \tfrac{1}{2}(e^x - e^{-x}),$$

prove
$$\frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\sinh x = \cosh x,$$

$\cosh 0 = 1$, $\sinh 0 = 0$; and hence expand $\cosh x$ and $\sinh x$ in powers of x. Verify that $\cosh x + \sinh x = e^x$, and $\cosh x - \sinh x = e^{-x}$. Compute $\cosh 2$ and $\sinh 2$ to four decimal places. Show that the error made in stopping the series at any term is much less than the last term used.

78. Taylor's series. If a function of the sum of two numbers a and x is given, $f(a+x)$, it is frequently desirable to expand the function in powers of one of them, say x.

In the function $f(a+x)$, a is to be regarded as constant, so that, considered as a function of x, it may be expanded by formula (3) of the preceding article. In that formula, the constant term in the expansion is the value which the function has when x is made equal to zero, hence the first term in the expansion of $f(a+x)$ may be written $f(a)$. In the same manner the coefficients of the successive powers of x are the corresponding derivatives of $f(a+x)$ as to x, in which x is put equal to zero after the differentiation has been performed. The expansion may therefore be written

$$f(a+x) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \cdots + \frac{f^{(n)}(a)}{n!}x^n + \cdots.$$

This series, from the name of its discoverer, is known as *Taylor's series*, and the theorem expressed by the formula is known as Taylor's theorem.

Ex. **1.** Expand $\sin(a+x)$ in powers of x.

Here $$f(a+x) = \sin(a+x),$$
hence $$f(a) = \sin a,$$
and $$f'(a) = \cos a,$$
$$\cdots\cdots\cdots$$

Hence $$\sin(a+x) = \sin a + \cos a \cdot x - \frac{\sin a}{2!}x^2 - \frac{\cos a}{3!}x^3 + \cdots.$$

Ex. **2.** Compute $\sin 61°$, by putting $a = 60°$.

EXERCISES

1. Expand $\tan x$ in powers of x. Obtain three terms.

2. Compare the expansion of $\tan x$ with the quotient derived by dividing the series for $\sin x$ by that for $\cos x$.

3. Find a limit for the error which occurs in replacing $\cos x$ by the first three terms of its expansion in powers of x when $x = \frac{1}{3}$ of a radian.

4. Prove that $\log(x + \sqrt{1+x^2}) = x - \frac{x^3}{2 \cdot 3} + \frac{3}{2 \cdot 4} \cdot \frac{x^5}{5} - \cdots$.

5. Prove $\log \cos x = -\frac{x^2}{2} - \frac{2x^4}{4!} - \frac{16x^6}{6!} - \frac{272x^8}{8!} \cdots$.

6. Compute $\sin 1°$ correct to six places of decimals.

7. Expand $\sqrt{1-x^2}$ in powers of x, and compare with the expansion by the binomial theorem.

8. Expand $\cos x$ in powers of $x - \frac{\pi}{6}$.

9. Expand e^{x+h} in powers of h.

10. Arrange $3x^3 - 5x^2 + 8x - 5$ in powers of $x - 2$.

11. Expand $\log(x+h)$ in powers of h.

12. Arrange $x^4 - 1$ in powers of $x + 1$.

13. Prove the binomial theorem $(a+x)^n = a^n + \cdots = \sum_{r=0}^{n} \frac{n!\, a^{n-r}x^r}{r!\,(n-r)!}$.

Find the form of the series when n is not an integer, and determine the interval of convergence.

14. Find $\sqrt[3]{126} = \sqrt[3]{125+1} = 5\sqrt[3]{1+\frac{1}{125}}$ to three places of decimals by the binomial theorem.

15. Find $\sqrt[3]{1334}$.

16. Calculate log 31.

17. What is the greatest value of m that will permit the approximation $(1+m)^4 = 1 + 4m$ with an error not exceeding .001?

18. Expand $\frac{1}{x}$ in powers of $x-1$ and find the interval of convergence.

79. Rolle's theorem. By Art. 76 a series can be the correct representation of its generating function only when the remainder after n terms can be made as small as desired by taking n large enough. Before obtaining the form of this remainder it is necessary to introduce the following lemma.

Rolle's theorem. If $f(x)$ and its first derivative are continuous for all values of x between a and b, and if $f(a)$, $f(b)$ both vanish, then $f'(x)$ will vanish for some value of x between a and b.

The proof follows immediately from the theorem of mean value (Art. 39). See Figure 41.

80. Form of remainder in Maclaurin's series. Let the remainder after n terms be denoted by $R_n(x, a)$, which is a function of x and a as well as of n. Since each of the succeeding terms is divisible by $(x-a)^n$, R_n may be conveniently written in the form

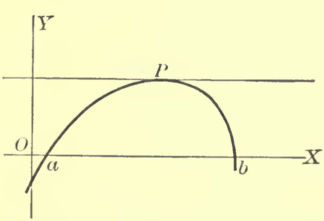

FIG. 41

$$R_n(x, a) = \frac{(x-a)^n}{n!}\phi(x, a).$$

The problem is now to determine $\phi(x, a)$ so that the relation

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots$$
$$+\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}+\frac{\phi(x, a)}{n!}(x-a)^n \qquad (1)$$

may be an algebraic identity, in which the right-hand member contains only the first n terms of the series, with the remainder after n terms. Thus, by transposing,

$$f(x)-f(a)-f'(a)(x-a)-\frac{f''(a)}{2!}(x-a)^2-\cdots$$
$$-\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}-\frac{\phi(x, a)}{n!}(x-a)^n\equiv 0. \qquad (2)$$

Let a new function, $F(z)$, be defined as follows:

$$F(z)\equiv f(x)-f(z)-f'(z)(x-z)-\frac{f''(z)}{2!}(x-z)^2-\cdots$$
$$-\frac{f^{(n-1)}(z)}{(n-1)!}(x-z)^{n-1}-\frac{\phi(x, a)}{n!}(x-z)^n. \qquad (3)$$

This function $F(z)$ vanishes when $z=x$, as is seen by inspection, and it also vanishes when $z=a$, since it then becomes identical with the left-hand member of (2); hence, by Rolle's theorem, its derivative $F'(z)$ vanishes for some value of z between x and a, say z_1. But

$$F'(z)=-f'(z)+f'(z)-f''(z)(x-z)+f''(z)(x-z)-\cdots$$
$$-\frac{f^{(n)}(z)}{(n-1)!}(x-z)^{n-1}+\frac{\phi(x, a)}{(n-1)!}(x-z)^{n-1}.$$

These terms cancel each other in pairs except the last two; hence

$$F'(z) = \frac{(x-z)^{n-1}}{(n-1)!}[\phi(x, a) - f^{(n)}(z)].$$

Since $F'(z)$ vanishes when $z = z_1$, it follows that

$$\phi(x, a) = f^{(n)}(z_1). \tag{4}$$

In this expression z_1 lies between x and a, and may thus be represented by

$$z_1 = a + \theta(x - a),$$

where θ is a positive proper fraction. Hence from (4)

$$\phi(x, a) = f^{(n)}[a + \theta(x - a)],$$

and

$$R_n(x, a) = \frac{f^{(n)}[a + \theta(x - a)]}{n!}(x - a)^n.*$$

The complete form of the expansion of $f(x)$ is then

$$\begin{aligned} f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(a + \theta(x-a))}{n!}(x-a)^n, \end{aligned} \tag{5}$$

in which n is any positive integer. The series may be carried to any desired number of terms by increasing n, and the last term in (5) gives the remainder (or error) after the first n terms of the series. The symbol $f^{(n)}(a + \theta(x - a))$ indicates that $f(x)$ is to be differentiated n times with regard to x, and that x is then to be replaced by $a + \theta(x - a)$.

* This form of the remainder was found by Lagrange (1736–1813), who published it in the Mémoires de l'Académie des Sciences à Berlin, 1772.

81. Another expression for the remainder. Instead of putting $R_n(x, a)$ in the form $\dfrac{(x-a)^n}{n!}\phi(x, a)$,

it is sometimes convenient to write it

$$R_n(x, a) = (x-a)\psi(x, a).$$

Proceeding as before, we find the expression for $F'(z)$,

$$F'(z) = -\frac{f^{(n)}(z)}{(n-1)!}(x-z)^{n-1} + \psi(x, a).$$

In order for this to vanish when $z = z_1$, it is necessary that

$$\psi(x, a) = \frac{f^{(n)}(z_1)}{(n-1)!}(x-z_1)^{n-1},$$

in which $\quad z_1 = a + \theta(x-a),\ x - z_1 = (x-a)(1-\theta).$

Hence $\quad \psi(x, a) = \dfrac{f^{(n)}(a+\theta(x-a))}{(n-1)!}(1-\theta)^{n-1}(x-a)^{n-1},$

and $\quad R_n(x, a) = (1-\theta)^{n-1}\dfrac{f^{(n)}(a+\theta(x-a))}{(n-1)!}(x-a)^n.$*

An example of the use of this form of remainder is furnished by the series for log x in powers of $x-a$, when $x-a$ is negative, and also in the expansion of $(a+x)^m$.

Ex. **1.** Find the interval of equivalence for the development of log x in powers of $x-a$, when a is a positive number.

Here, from Art. 77, Ex. 1,

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n},$$

hence $\quad f^{(n)}(a+\theta(x-a)) = (-1)^{n-1}\dfrac{(n-1)!}{(a+\theta(x-a))^n},$

and, by Art. 80,

$$R_n(x, a) = \frac{(-1)^{n-1}(x-a)^n}{n(a+\theta(x-a))^n} = \frac{(-1)^{n-1}}{n}\left[\frac{x-a}{a+\theta(x-a)}\right]^n.$$

* This form of the remainder was found by Cauchy (1789–1857), and first published in his "Leçons sur le calcul infinitésimal," 1826.

First let $x - a$ be positive. Then when it lies between 0 and a, it is numerically less than $a + \theta(x - a)$, since θ is a positive proper fraction; hence when $n \doteq \infty$

$$\left[\frac{x-a}{a+\theta(x-a)}\right]^n \doteq 0, \text{ and } R_n(x, a) \doteq 0.$$

Again, when $x - a$ is negative and numerically less than a, the second form of the remainder must be employed. As before

$$f^{(n)}(a + \theta(x-a)) = \frac{(-1)^{n-1}(n-1)!}{(a+\theta(x-a))^n},$$

hence
$$R_n(x, a) = (1-\theta)^{n-1} \cdot \frac{(-1)^{n-1}(x-a)^n}{[a+\theta(x-a)]^n}$$

$$= (1-\theta)^{n-1} \cdot \frac{-(a-x)^n}{[a-\theta(a-x)]^n}$$

$$= -\left[\frac{(a-x)-\theta(a-x)}{a-\theta(a-x)}\right]^{n-1} \cdot \frac{a-x}{a-\theta(a-x)}.$$

The factor within the brackets is numerically less than 1, hence the $(n-1)$th power can be made less than any given number, by taking n large enough. This is true for all values of x between 0 and a.

Therefore, $\log x$ and its development in powers of $x - a$ are equivalent within the interval of convergence of the series, that is, for all values of x between 0 and $2a$.

Ex. **2.** Show that the development of $x^{-\frac{1}{2}}$ in positive powers of $x - a$ holds for all values of x that make the series convergent; that is, when x lies between 0 and $2a$.

If the function is expanded in powers of x, the complete form will be

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$
$$+ \frac{f^{(n)}(\theta x)}{n!}x^n \tag{1}$$

for the first form of remainder, and

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

$$+\frac{f^{(n)}(\theta x)}{(n-1)!}(1-\theta)^{n-1}\cdot x^n \tag{2}$$

for the second form of remainder.

Similarly, the complete form of Taylor's series (Art. 78) becomes

$$f(a+x)=f(a)+f'(a)x+\frac{f''(a)}{2!}x^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}x^{n-1}$$

$$+\frac{f^{(n)}(a+\theta x)}{n!}x^n \tag{3}$$

for the first form of remainder, and

$$f(a+x)=f(a)+f'(a)x+\frac{f''(a)}{2!}x^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}x^{n-1}$$

$$+\frac{f^{(n)}(a+\theta x)}{(n-1)!}(1-\theta)^{n-1}\cdot x^n \tag{4}$$

for the second form of remainder.

These forms are of no value for numerical computation unless $f^{(n)}(x)$ can be determined, but may sometimes be used to advantage to obtain a maximum error, corresponding to small values of n. It should be observed that when $n=1$, the theorem of mean value results. (Art. 39.)

Ex. **3**. Obtain the limit of error in retaining but two non-vanishing terms in the expansion of $\log(x+\sqrt{1+x^2})$ when $x=\frac{1}{4}$.

$$\log(x+\sqrt{1+x^2})=x-\frac{x^3}{2\cdot 3}+\left[\frac{9y-6y^3}{(1+y^2)^{\frac{7}{2}}}\right]\frac{x^4}{24},$$

wherein $y=\theta x$.

The next step is to obtain the largest and the smallest value which the expression in brackets assumes for values of y within the interval 0 to $\frac{1}{4}$. For this purpose, consider the function

$$u = \frac{3\,y(3 - 2\,y^2)}{(1 + y^2)^{\frac{7}{2}}}.$$

We find that $\dfrac{du}{dy}$ is positive for all values of y between $y = 0$ and $y = \frac{1}{4}$; hence u has its largest value when $y = \frac{1}{4}$, and the corresponding value of the last term is .000284.

Ex. **4**. How many terms should be used in the expansion of e^x in powers of x to insure a result correct to four places of decimals when $x = \frac{1}{2}$?

Ex. **5**. In the expansion of $\log_{10}(1 + x)$ in powers of x how many terms should be used in order to obtain the value of $\log_{10}(1.8)$ correct to 5 decimals?

CHAPTER XI

INDETERMINATE FORMS

82. Hitherto the values of a given function $f(x)$, corresponding to assigned values of the variable x, have been obtained by direct substitution. The function may, however, involve the variable in such a way that for certain values of the latter the corresponding values of the function cannot be found by mere substitution.

For example, the function

$$\frac{e^x - e^{-x}}{\sin x},$$

for the value $x = 0$, assumes the form $\frac{0}{0}$, and the corresponding value of the function is thus not directly determined. In such a case the expression for the function is said to assume an *indeterminate form* for the assigned value of the variable.

The example just given illustrates the indeterminateness of most frequent occurrence; namely, that in which the given function is the quotient of two other functions that vanish for the same value of the variable.

Thus if $$f(x) = \frac{\phi(x)}{\psi(x)},$$

and if, when x takes the special value a, the functions $\phi(x)$ and $\psi(x)$ both vanish, then

$$f(a) = \frac{\phi(a)}{\psi(a)} = \frac{0}{0}$$

is indeterminate in form, and cannot be rendered determinate without further transformation.

83. Indeterminate forms may have determinate values. A case has already been noticed (Art. 2) in which an expression that assumes the form $\frac{0}{0}$ for a certain value of its variable takes a definite value, dependent upon the law of variation of the function in the vicinity of the assigned value of the variable.

As another example, consider the function

$$y = \frac{x^2 - a^2}{x - a}.$$

If this relation between x and y is written in the forms

$$y(x-a) = x^2 - a^2, \quad (x-a)(y-x-a) = 0,$$

it will be seen that it can be represented graphically, as in Fig. 42, by the pair of lines

$$x - a = 0,$$
$$y - x - a = 0.$$

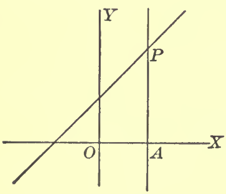

FIG. 42

Hence when x has the value of a, there is an indefinite number of corresponding points on the locus, all situated on the line $x = a$; and accordingly for this value of x the function y may have any value whatever, and is therefore indeterminate.

When x has any value different from a, the corresponding value of y is determined from the equation $y = x + a$. Now, of the infinite number of different values of y corresponding to $x = a$, there is one particular value AP which is continuous with the series of values taken by y when x takes successive values in the vicinity of $x = a$. This may be called the *determinate value* of y when $x = a$. It is obtained by putting $x = a$ in the equation $y = x + a$, and is therefore $y = 2a$.

This result may be stated without reference to a locus as follows. When $x = a$, the function

$$\frac{x^2 - a^2}{x - a}$$

is indeterminate, and has an infinite number of different values; but among these values there is one determinate value which is continuous with the series of values taken by the function as x increases through the value a; this determinate or singular value may then be defined by

$$\lim_{x \doteq a} \frac{x^2 - a^2}{x - a}.$$

In evaluating this limit the common factor $x - a$ may be removed from numerator and denominator, since this factor is not zero while x is different from a; hence the determinate value of the function is

$$\lim_{x \doteq a} \frac{x + a}{1} = 2a.$$

Ex. **1.** Find the determinate value, when $x \doteq 1$, of the function

$$\frac{x^3 + 2x^2 - 3x}{3x^3 - 3x^2 - x + 1},$$

which, at the limit, takes the form $\frac{0}{0}$.

This expression may be written in the form

$$\frac{(x^2 + 3x)(x - 1)}{(3x^2 - 1)(x - 1)},$$

which reduces to $\frac{x^2 + 3x}{3x^2 - 1}$. When $x = 1$, this becomes $\frac{4}{2} = 2$.

Ex. **2.** Evaluate the expression

$$\frac{x^3 + ax^2 + a^2x + a^3}{x^3 + b^2x + ax^2 + ab^2}$$

when $x \doteq -a$.

Ex. **3.** Determine the value of

$$\frac{x^3 - 7x^2 + 3x + 14}{x^3 + 3x^2 - 17x + 14}$$

when $x \doteq 2$.

Ex. **4.** Evaluate $\dfrac{a - \sqrt{a^2 - x^2}}{x^2}$ when $x \doteq 0$.

(Multiply both numerator and denominator by $a + \sqrt{a^2 - x^2}$.)

84. Evaluation by development. In some cases the common vanishing factor can be best removed after expansion in series.

Ex. **1.** Consider the function mentioned in Art. 82,

$$\frac{e^x - e^{-x}}{\sin x}.$$

When numerator and denominator are developed in powers of x, the expression becomes

$$\frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)}{x - \frac{x^3}{3!} + \cdots}$$

$$= \frac{2x + \frac{2}{3!}x^3 + \cdots}{x - \frac{x^3}{3!} + \cdots} = \frac{2 + \frac{x^2}{3} + \cdots}{1 - \frac{x^2}{6} + \cdots},$$

which has the determinate value 2, when x takes the value zero.

Ex. **2.** As another example, evaluate, when $x \doteq 0$, the function

$$\frac{x - \sin^{-1} x}{\sin^3 x}.$$

By development it becomes

$$\frac{x - \left(x + \frac{1}{2} \cdot \frac{x^3}{3} + \cdots\right)}{\left(x - \frac{x^3}{3!} + \cdots\right)^3} = \frac{-\frac{x^3}{6} + \cdots}{x^3 + \cdots}.$$

Removing the common factor, and then putting $x = 0$, we obtain $-\frac{1}{6}$.

In these two examples the assigned value of x, for which the indeterminateness occurs, is zero, and the developments are made in powers of x. If the assigned value of x is some other number, as a, then the development should be made in powers of $x - a$.

Ex. 3. Evaluate, when $x = \frac{\pi}{2}$, the function

$$\frac{\cos x}{1 - \sin x}.$$

By putting $x - \frac{\pi}{2} = h$, $x = \frac{\pi}{2} + h$, the expression becomes

$$\frac{\cos\left(\frac{\pi}{2} + h\right)}{1 - \sin\left(\frac{\pi}{2} + h\right)} = \frac{-\sin h}{1 - \cos h} = \frac{-h + \frac{h^3}{6} - \cdots}{\frac{h^2}{2} - \frac{h^4}{24} + \cdots} = \frac{-1 + \frac{h^2}{6} - \cdots}{\frac{h}{2} - \frac{h^3}{24} + \cdots},$$

which becomes infinite when $h = 0$; that is, when $x = \frac{\pi}{2}$.

Hence $$\lim_{x \doteq \frac{\pi}{2}} \frac{\cos x}{1 - \sin x} = \pm \infty,$$

according as h approaches zero from the negative or positive side.

85. Evaluation by Differentiation. Let the given function be of the form $\frac{f(x)}{\phi(x)}$, and suppose that $f(a) = 0$, $\phi(a) = 0$. It is required to find $\lim\limits_{x \doteq a} \frac{f(x)}{\phi(x)}$.

We assume that $f(x)$, $\phi(x)$ can be developed in the vicinity of $x = a$, by expanding them in powers of $x - a$. Then

$$\frac{f(x)}{\phi(x)} = \frac{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots}{\phi(a) + \phi'(a)(x-a) + \frac{\phi''(a)}{2!}(x-a)^2 + \cdots}$$

$$= \frac{f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots}{\phi'(a)(x-a) + \frac{\phi''(a)}{2!}(x-a)^2 + \cdots}.$$

By dividing by $x-a$ and then letting $x \doteq a$, we obtain

$$\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}.$$

By hypothesis the functions $f'(a)$, $\phi'(a)$ will both be finite.

If $f'(a)=0$, $\phi'(a) \neq 0$, then $\dfrac{f(a)}{\phi(a)} = 0$.

If $f'(a) \neq 0$, $\phi'(a) = 0$, then $\dfrac{f(a)}{\phi(a)} = \infty$.

If $f'(a)$ and $\phi'(a)$ are both zero, the limiting value of $\dfrac{f(x)}{\phi(x)}$ is to be obtained by carrying Taylor's development one term farther, removing the common factor $(x-a)^2$, and then letting x approach a. The result is $\dfrac{f''(a)}{\phi''(a)}$.

Similarly, if $f(a)$, $f'(a)$, $f''(a)$; $\phi(a)$, $\phi'(a)$, $\phi''(a)$ all vanish, it is proved in the same manner that

$$\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \frac{f'''(a)}{\phi'''(a)},$$

and so on, until a result is obtained that is not indeterminate in form.

Hence the rule:

To evaluate an expression of the form $\dfrac{0}{0}$, *differentiate numerator and denominator separately; substitute the critical value of x in their derivatives, and equate the quotient of the derivatives to the indeterminate form.*

Ex. **1.** Evaluate $\dfrac{1-\cos\theta}{\theta^2}$ when $\theta = 0$.

Put $\qquad f(\theta) = 1 - \cos\theta, \qquad \phi(\theta) = \theta^2.$

Then $\qquad f'(\theta) = \sin\theta, \qquad \phi'(\theta) = 2\theta,$

and $\qquad f'(0) = 0, \qquad \phi'(0) = 0.$

Thus, the function $\dfrac{f'(\theta)}{\phi'(\theta)}$ is also indeterminate at $\theta = 0$. It is therefore necessary to obtain $\dfrac{f''(0)}{\phi''(0)}$.

Accordingly, $f''(\theta) = \cos\theta, \qquad \phi''(\theta) = 2,$

$$f''(0) = 1, \qquad \phi''(0) = 2,$$

hence
$$\lim_{\theta \doteq 0} \frac{1-\cos\theta}{\theta^2} = \frac{1}{2}.$$

Ex. 2. Find $\displaystyle\lim_{x \doteq 0} \frac{e^x + e^{-x} + 2\cos x - 4}{x^4}$.

$$\lim_{x \doteq 0} \frac{e^x + e^{-x} + 2\cos x - 4}{x^4} = \lim_{x \doteq 0} \frac{e^x - e^{-x} - 2\sin x}{4x^3}$$

$$= \lim_{x \doteq 0} \frac{e^x + e^{-x} - 2\cos x}{12x^2}$$

$$= \lim_{x \doteq 0} \frac{e^x - e^{-x} + 2\sin x}{24x}$$

$$= \lim_{x \doteq 0} \frac{e^x + e^{-x} + 2\cos x}{24}$$

$$= \frac{1}{6}.$$

Ex. 3. Find $\displaystyle\lim_{x \doteq 0} \frac{x - \sin x \cos x}{x^3}$.

Ex. 4. Find $\displaystyle\lim_{x \doteq 1} \frac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}$.

In this example, show that $x - 1$ is a factor of both numerator and denominator.

Ex. 5. Find $\displaystyle\lim_{x \doteq 0} \frac{3\tan x - 3x - x^3}{x^5}$.

In applying this process to particular problems, the work can often be shortened by evaluating a non-vanishing factor in either numerator or denominator before performing the differentiation.

Ex. **6.** Find $\lim\limits_{x \doteq 0} \dfrac{(x-4)^2 \tan x}{x}$.

The given expression may be written

$$\lim_{x \doteq 0} (x-4)^2 \frac{\tan x}{x} = \lim_{x \doteq 0} (x-4)^2 \lim_{x \doteq 0} \frac{\tan x}{x}$$
$$= 16 \cdot 1 = 16.$$

In general, if $f(x) = \psi(x)\chi(x)$, and if $\psi(a) = 0$, $\chi(a) \neq 0$, $\phi(a) = 0$, then

$$\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \chi(a) \frac{\psi'(a)}{\phi'(a)}.$$

For

$$\lim_{x \doteq a} \frac{\psi(x)\chi(x)}{\phi(x)} = \lim_{x \doteq a} \chi(x) \cdot \lim_{x \doteq a} \frac{\psi(x)}{\phi(x)} = \chi\ a) \cdot \frac{\psi'(a)}{\phi'(a)}.$$

Ex. **7.** Find $\lim\limits_{x \doteq \frac{\pi}{2}} \dfrac{\sin x \cos^2 x}{(2x - \pi)^2}$.

Ex. **8.** Find $\lim\limits_{x \doteq 1} \dfrac{(x-3)^2 \log(2-x)}{\sin(x-1)}$.

EXERCISES

Evaluate the following expressions:

1. $\dfrac{1 - \cos x}{\sin x}$ when $x = 0$.

2. $\dfrac{e^x - e^{-x}}{\sin x}$ when $x = 0$.

3. $\dfrac{x^3 - 1}{x - 1}$ when $x = 1$.

4. $\dfrac{a^x - 1}{b^x - 1}$ when $x = 0$.

5. $\dfrac{\sin ax}{\sin bx}$ when $x = 0$.

6. $\dfrac{(1 + x)^{\frac{1}{n}} - 1}{x}$ when $x = 0$.

7. $\dfrac{e^x + e^{-x} - 2}{x^2}$ when $x = 0$.

8. $\dfrac{\tan x - \sin x \cos x}{x^3}$ when $x = 0$.

9. $\dfrac{\sin^{-1} x}{\tan^{-1} x}$ when $x = 0$.

10. $\dfrac{e^x \sin x - x - x^2}{x^2 + x \log(1 - x)}$ when $x = 0$.

11. $\dfrac{\tan x - \sin x}{x^3}$ when $x = 0$.

There are other indeterminate forms than $\dfrac{0}{0}\cdot$ They are $\dfrac{\infty}{\infty}$, $\infty - \infty$, 0^0, 1^∞, ∞^0.

86. Evaluation of the indeterminate form $\frac{\infty}{\infty}$.

Let the function $\dfrac{f(x)}{\phi(x)}$ become $\frac{\infty}{\infty}$ when $x=a$. It is required to find $\underset{x \doteq a}{\lim} \dfrac{f(x)}{\phi(x)}$.

This function can be written

$$\frac{f(x)}{\phi(x)}=\frac{\dfrac{1}{\phi(x)}}{\dfrac{1}{f(x)}},$$

which takes the form $\dfrac{0}{0}$ when $x=a$, and can therefore be evaluated by the preceding rule.

When $x \doteq a$,

$$\underset{x \doteq a}{\lim} \frac{f(x)}{\phi(x)}=\underset{x \doteq a}{\lim} \frac{\dfrac{1}{\phi(x)}}{\dfrac{1}{f(x)}}=\underset{x \doteq a}{\lim} \frac{-\dfrac{\phi'(x)}{[\phi(x)]^2}}{-\dfrac{f'(x)}{[f(x)]^2}}$$

$$=\underset{x \doteq a}{\lim} \left[\frac{f(x)}{\phi(x)}\right]^2 \frac{\phi'(x)}{f'(x)}. \tag{1}$$

If both members are divided by $\underset{x \doteq a}{\lim} \dfrac{f(x)}{\phi(x)}$, when this limit is not 0 nor ∞, the equation becomes

$$1=\underset{x \doteq a}{\lim} \frac{f(x)}{\phi(x)} \frac{\phi'(x)}{f'(x)},$$

therefore

$$\underset{x \doteq a}{\lim} \left[\frac{f(x)}{\phi(x)}\right]=\frac{f'(a)}{\phi'(a)}. \tag{2}$$

This is exactly the same result as was obtained for the form $\dfrac{0}{0}$; hence the procedure for evaluating the indeterminate forms $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, is the same in both cases.

When the true value of $\frac{f(a)}{\phi(a)}$ is 0 or ∞, equation (1) is satisfied, independent of the value of $\frac{f'(a)}{\phi'(a)}$; but (2) still gives the correct value. For, suppose $\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = 0$. Consider the function

$$\frac{f(x)}{\phi(x)} + c = \frac{f(x) + c\phi(x)}{\phi(x)},$$

which has the form $\frac{\infty}{\infty}$ when $x = a$, and has the determinate value c, which is not zero. Hence by (2)

$$\lim_{x \doteq a} \frac{f(x) + c\phi(x)}{\phi(x)} = \frac{f'(a) + c\phi'(a)}{\phi'(a)} = \frac{f'(a)}{\phi'(a)} + c.$$

Therefore, by subtracting c,

$$\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}.$$

If $\lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \infty$, then $\lim_{x \doteq a} \frac{\phi(x)}{f(x)} = 0$, which can be treated as the previous case.

The forms $0 \cdot \infty$ and $\infty - \infty$ can usually be evaluated by putting them in one or the other of the forms already discussed. In the case of the others, in which the indeterminateness appears in the exponent, the logarithm of the function can be reduced to one of the preceding forms.

EXERCISES

Evaluate the following expressions:

1. $\frac{\log \sin 2x}{\log \sin x}$ when $x = 0$.

2. $\frac{\log x}{\cot x}$ when $x = 0$.

3. $\frac{x^n}{e^x}$ when $x = \infty$.

4. $\frac{\tan x}{\tan 5x}$ when $x = \frac{\pi}{2}$.

5. $\frac{\sec 3x}{\sec 5x}$ when $x = \frac{\pi}{2}$.

6. $x^{\sin x}$ when $x = 0$.

7. $(\cos ax)^{\csc^2 cx}$ when $x = 0$.

CHAPTER XII

CONTACT AND CURVATURE

87. Order of contact. The points of intersection of the two curves

$$y=\phi(x), \quad y=\psi(x)$$

are found by making the two equations simultaneous; that is, by finding those values of x for which

$$\phi(x)=\psi(x).$$

Suppose $x=a$ is one value that satisfies this equation. Then the point $x=a$, $y=\phi(a)=\psi(a)$ is common to the curves.

If, moreover, the two curves have the same tangent at this point, they are said to touch each other, or to have *contact* with each other. The values of y and of $\frac{dy}{dx}$ are thus the same for both curves at the point in question, which requires that

$$\phi(a)=\psi(a),$$
$$\phi'(a)=\psi'(a).$$

If, in addition, the value of $\frac{d^2y}{dx^2}$ is the same for each curve at the point, then

$$\phi''(a)=\psi''(a),$$

and the curves are said to have a *contact of the second order* with each other, provided $\phi'''(a) \neq \psi'''(a)$.

If $\phi(a)=\psi(a)$, and all the derivatives up to the nth order inclusive are equal to each other, that is, $\phi'(a)=\psi'(a)$, $\phi''(a)=\psi''(a)$, $\cdots$, $\phi^{(n)}(a)=\psi^{(n)}(a)$, but $\phi^{(n+1)}(a)\neq\psi^{(n+1)}(a)$, the curves are said to have contact of the nth order.

88. Number of conditions implied by contact. or e equation of the curve $y=\phi(x)$ is given, and it is requir to determine the equation of a second curve $y=\psi(x)$ that shall have contact of any given order with $y=\phi(x)$ at a specified point, then, from the definition given in the preceding article for contact of the nth order, $n+1$ conditions must be imposed upon the coefficients in $\psi(x)$. The required curve must therefore contain at least $n+1$ arbitrary constants in order to fulfill the required conditions.

A straight line has two arbitrary constants, which can be determined by two conditions; accordingly a straight line can be drawn which touches a given curve at any specified point. For if the equation of a line is written $y=mx+b$, then

$$\frac{dy}{dx}=m,\quad \frac{d^2y}{dx^2}=0;$$

hence, through any arbitrary point $x=a$ on a given curve $y=\phi(x)$, a line can be drawn which has contact of the first order with the curve, but which has not in general contact of the second order; for the two conditions for first-order contact are

$$ma+b=\phi(a),$$
$$m=\phi'(a),$$

which are just sufficient to determine m and b.

In general no line can be drawn having contact of an order higher than the first with a given curve at a given point; but there are certain special points at which this can be done. For example, the additional condition for second-order contact is $0=\phi''(a)$, which is satisfied when the point $x=a$ is a point of inflexion on the given curve $y=\phi(x)$. (Art. 49.) Thus the tangent at a point of inflexion on a curve has contact of the second order with the curve.

The equation of a circle has three independent constants. It is therefore possible to determine a circle having contact of the second order with a given curve at any assigned point.

The equation of a parabola has four constants, hence a parabola can be found which has contact of the third order with the given curve at a given point.

The general equation of a central conic has five independent constants, hence a conic can be found which has contact of the fourth order with a given curve at any specified point.

As in the case of the tangent line, special points may be found for which these curves have contact of higher order.

89. Contact of odd and of even order.

THEOREM. At a point where two curves have contact of an odd order they do not cross each other; but they do cross where they have contact of an even order.

For, let the curves $y = \phi(x)$, $y=\psi(x)$ have contact of the nth order at the point whose abscissa is a; and let y_1, y_2 be the ordinates of these curves at the point whose abscissa is $a + h$.

Then
$$y_1 = \phi(a + h), \quad y_2 = \psi(a + h),$$

and by Taylor's theorem

$$y_1 = \phi(a) + \phi'(a) \cdot h + \frac{\phi''(a)}{2!} \cdot h^2 + \cdots$$
$$+ \frac{\phi^n(a)}{n!} \cdot h^n + \frac{h^{n+1}}{(n+1)!} \phi^{n+1}(a) + \cdots;$$

$$y_2 = \psi(a) + \psi'(a) \cdot h + \frac{\psi''(a)}{2!} \cdot h^2 + \cdots$$
$$+ \frac{\psi^n(a)}{n!} \cdot h^n + \frac{h^{n+1}}{(n+1)!} \cdot \psi^{n+1}(a) + \cdots.$$

Since by hypothesis the two curves have contact of the nth order at the point whose abscissa is a, hence

$$\phi(a)=\psi(a),\ \ \phi'(a)=\psi'\ a),\ \cdots,\ \ \phi^n(a)=\psi^n(a),$$

and $$y_1-y_2=\frac{h^{n+1}}{(n+1)!}[\phi^{n+1}(a)+\ \cdots\ -\psi^{n+1}(a)-\ \cdots];$$

but this expression, when h is sufficiently diminished, has the same sign as

$$h^{n+1}[\phi^{n+1}(a\ -\psi^{n+1}(a)].$$

Hence, if n is odd, y_1-y_2 does not change sign when h is changed into $-h$, and thus the two curves do not cross each other at the common point. On the other hand, if n is even, y_1-y_2 changes sign with h; and therefore when the contact is of even order the curves cross each other at their common point.

Geometrically, we may say that two curves having contact of the nth order pass through $n+1$ common points which approach coincidence at the point of contact. For let $y=\phi(x)$, $y=\psi(x)$ touch each other at $x=a$. This means that they have two coincident points in common at $(a, \phi(a))$, and the conditions to be satisfied are

$$\phi(a)=\psi(a),\ \ \phi'(a)=\psi'(a).$$

If the curves also have a point in common for $x=a+h$, then

$$\phi(a+h)=\psi(a+h).$$

Expanding by Taylor's series and making use of the preceding conditions, we may cancel the common factor h^2. If now this condition is still satisfied when h approaches zero, so that the third point of intersection approaches the position of the two coincident ones, then we must have the further condition

$\phi''(a) = \psi''(a)$. Thus, three coincident points of intersection imply contact of the second order. By repeating this argument the above theorem results.

For example, the tangent line usually lies entirely on one side of the curve, but at a point of inflexion the tangent crosses the curve.

Again, the circle of second-order contact crosses the curve except at the special points noted later, in which the circle has contact of the third order.

EXERCISES

1. Find the order of contact of the curves

$$4y = x^2 \text{ and } y = x - 1.$$

2. Find the order of contact of the curves

$$x = y^3 \text{ and } x - 2y + 1 = 0.$$

3. Find the order of contact of the curves

$$4y = x^2 - 4 \text{ and } x^2 - 2y = 3 - y^2.$$

4. Determine the parabola having its axis parallel to the y-axis, which has the closest possible contact with the curve $a^2y = x^3$ at the point (a, a). (The equation of a parabola having its axis parallel to the y-axis is of the form

$$y = Ax^2 + Bx + C.)$$

5. Determine a straight line which has contact of the second order with the curve

$$y = x^3 - 3x^2 - 9x + 9.$$

6. Find the order of contact of

$$y = \log(x - 1) \text{ and } x^2 - 6x + 2y + 8 = 0$$

at the point (2, 0).

7. What must be the value of a in order that the curves

$$y = x + 1 + a(x - 1)^2 \text{ and } xy = 3x - 1$$

may have contact of the second order?

8. Determine the parabola which has its axis parallel to the y-axis and has contact of the second order with the hyperbola $xy = 1$ at the point (1, 1).

9. Determine the point and order of contact of the curves

$$(a)\ y = x^3,\ y = 6x^2 - 9x + 4;$$
$$(b)\ y = x^3,\ y = -6x^2 - 12x - 8.$$

10. Determine the parabola which has its axis parallel to the y-axis, passes through the point (0, 3), and has contact of the first order with the curve $y = 2x^2$ at the point (1, 2). Similarly for a parabola having its axis parallel to the x-axis.

11. Show that the curve $y = \sin x$ has contact of the sixth order with the curve

$$y = x - \frac{x^3}{6} + \frac{x^5}{120}$$

at the origin. Show that $y = \sin x$, $y = \sinh x$, have contact of the second order at the origin. Draw these curves.

12. Find the order of contact of the curves $y = \cos x$, $y = \cosh x$ at the point (0, 1). Sketch the curves.

13. Find the order of contact at the origin of the curves

$$y = \tan x,\ y = \tanh x \equiv \frac{\sinh x}{\cosh x}.$$

90. Circle of curvature. The circle that has contact of the closest order with a given curve at a specified point is called the *osculating circle* or *circle of curvature* of the curve at the given point. The radius of this circle is called the *radius of curvature*, and its center is called the *center of curvature* at the assigned point.

91. Length of radius of curvature; coördinates of center of curvature. Let the equation of a circle be

$$(X - \alpha)^2 + (Y - \beta)^2 = R^2, \tag{1}$$

in which R is the radius, and α, β are the coördinates of the center, the current coördinates being denoted by X, Y to distinguish them from the coördinates of a point on the given curve.

It is required to determine R, α, β, so that this circle may have contact of the second order with the given curve at the point (x, y).

From (1), by successive differentiation, we deduce

$$\left.\begin{aligned} (X-\alpha)+(Y-\beta)\frac{dY}{dX}&=0,\\ 1+\left(\frac{dY}{dX}\right)^2+(Y-\beta)\frac{d^2Y}{dX^2}&=0. \end{aligned}\right\} \tag{2}$$

If the circle (1) has contact of the second order at the point (x, y) with the given curve, then when $X=x$ it is necessary that

$$\left.\begin{aligned} Y&=y,\\ \frac{dY}{dX}=\frac{dy}{dx},\quad \frac{d^2Y}{dX^2}&=\frac{d^2y}{dx^2}. \end{aligned}\right\} \tag{3}$$

Substituting these expressions in (2), we obtain

$$\left.\begin{aligned} (x-\alpha)+(y-\beta)\frac{dy}{dx}&=0,\\ 1+\left(\frac{dy}{dx}\right)^2+(y-\beta)\frac{d^2y}{dx^2}&=0, \end{aligned}\right\} \tag{4}$$

whence

$$y-\beta=-\frac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}},\quad x-\alpha=\frac{\dfrac{dy}{dx}\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}{\dfrac{d^2y}{dx^2}}, \tag{5}$$

and finally, by substitution in (1),

$$R=\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}. \tag{6}$$

Ex. 1. For the curve $y = \sin x$, show that $\alpha = x + \cot x(1 + \cos^2 x)$, $\beta = -2 \cos x \csc x$, $R = -(1 + \cos^2 x)^{\frac{3}{2}} \csc x$. Find the numerical values of α and β when $x = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$, and locate the corresponding points (α, β) on a drawing. Sketch roughly the path of this point as x varies. Write the equation of the osculating circle for the point $x = \frac{\pi}{3}$, and for $x = \frac{\pi}{2}$. Draw these circles.

Ex. 2. For the curve $y = x^3$, find α, β, R in terms of x. Compute their numerical values at $x = 1, .7, .5, .3, 0$. Show that R is a minimum when $x = \frac{1}{\sqrt[4]{45}} = .39 \cdots$, and that the value of R is $.57 \cdots$.

92. Limiting intersection of normals. Let $P \equiv (x_1, y_1)$ and $P' \equiv (x_2, y_2)$ be two points on a given curve $f(x, y) = 0$. The equations of the normals at these points are

$$(x - x_1) + (y - y_1)\frac{dy_1}{dx_1} = 0,$$

$$(x - x_2) + (y - y_2)\frac{dy_2}{dx_2} = 0.$$

If (α', β') is the point of intersection of these two lines, then

$$(\alpha' - x_1) + (\beta' - y_1)\frac{dy_1}{dx_1} = 0,$$

$$(\alpha' - x_2) + (\beta' - y_2)\frac{dy_2}{dx_2} = 0.$$

Now consider the function $\psi(x)$ of x defined by the equations

$$\psi(x) = (x - \alpha') + (y - \beta')\frac{dy}{dx}, \qquad f(x, y) = 0.$$

Since $\psi(x_1) = 0$ and $\psi(x_2) = 0$, hence by Rolle's theorem (Art. 79) it follows that

$$\psi'(\bar{x}) = 0,$$

in which $\bar{x}$ is defined by the inequalities

$$x_1 < \bar{x} < x_2.$$

Hence α', β' may be determined by the simultaneous equations

$$\psi(x_1) = 0, \quad \psi'(\bar{x}) = 0.$$

When $P' \equiv (x_2, y_2)$ approaches coincidence with the point $P \equiv (x_1, y_1)$, then $\psi'(x) \doteq \psi'(x_1)$, and therefore from (4), the point (α', β') becomes the center of curvature, hence:

The center of curvature at a point P on a curve is the limiting position of the point of intersection of the normal at P with the normal at the point P', when P' approaches the position of P.

93. Direction of radius of curvature. Since, at any point P on the given curve, the value of $\dfrac{dy}{dx}$ is the same for the curve and the osculating circle for that point, it follows that they have the same tangent and normal at P, and hence that the radius of curvature coincides with the normal. Again, since the value of $\dfrac{d^2y}{dx^2}$ is the same for both curves at P, it follows from Art. 50, that they have the same direction of bending at that point, and hence that the center of curvature lies on the concave side of the given curve (Fig. 43).

It follows from this fact and Art. 87 that the osculating circle is the limiting position of a circle passing through three points on the curve when these points move into coincidence.

The radius of curvature is usually regarded as positive or negative according as the bending of the curve is positive or negative (Art. 49), that is, according as the value of $\dfrac{d^2y}{dx^2}$ is positive or negative; hence, in the expression for R, the radical in the numerator is always to be given the positive sign.

The sign of R changes as the point P passes through a point of inflexion on the given curve (Fig. 44). It is evident from the figure that in this case R passes through an infinite value;

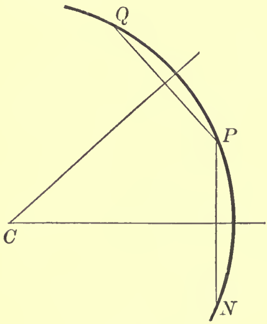

FIG. 43

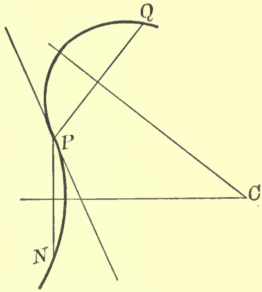

FIG. 44

for the circle through the points N, P, Q approaches coincidence with the inflexional tangent when N and Q approach coincidence with P, and the center of this circle at the same time passes to infinity.

94. Total curvature of a given arc; average curvature. The total curvature of an arc PQ (Fig. 45) in which the bending is in one direction, is the angle through which the tangent swings as the point of contact moves from the initial point P to the terminal point Q; or, in other words, it is the angle between the tangents at P and Q, measured from the former to the latter. Thus the total curvature of a given arc is positive or negative according as the bending is in the positive or negative direction of rotation.

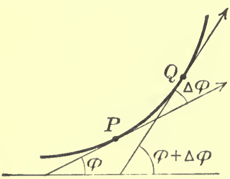

FIG. 45

The total curvature of an arc divided by the length of the arc is called the *average* curvature of the arc. If the length of the arc PQ is Δs centimeters, and if its total curvature is $\Delta\phi$ radians, then its average curvature is $\frac{\Delta\phi}{\Delta s}$ radians per centimeter.

95. Measure of curvature at a given point. The *measure of the curvature* of a given curve at a given point P is the limit which the average curvature of the arc PQ approaches when the point Q approaches coincidence with P.

Since the average curvature of the arc PQ is $\frac{\Delta\phi}{\Delta s}$, the measure of the curvature at the point P is

$$\kappa = \lim_{\Delta s \doteq 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds},$$

and may be regarded as the rate of deflection of the arc from the tangent estimated per unit of length; or, as the ratio of the angular velocity of the tangent to the linear velocity of the point of contact.

To express κ in terms of x, y, and the derivatives of y, we observe that

$$\tan\phi = \frac{dy}{dx};$$

whence

$$\phi = \tan^{-1}\frac{dy}{dx},$$

and

$$\frac{d\phi}{ds} = \frac{d}{ds}\left(\tan^{-1}\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\tan^{-1}\frac{dy}{dx}\right)\cdot\frac{dx}{ds}$$

$$= \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}\cdot\frac{1}{\frac{ds}{dx}};$$

therefore

$$\kappa = \frac{d\phi}{ds} = \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}. \qquad \text{[Art. 41.}$$

96. Curvature of an arc of a circle. In the case of a circular arc the normals are radii;

hence $$\Delta s = r \cdot \Delta \phi,\ \frac{\Delta \phi}{\Delta s} = \frac{1}{r}, \tag{1}$$

and therefore $$\kappa = \frac{1}{r}.$$

It follows that the average curvature of all arcs of the same circle is constant and equal to $\frac{1}{r}$ radians per unit of length.

For example, in a circle of 2 feet radius the total curvature of an arc of 3 feet is $\frac{3}{2} = 1.5$ radians, and the average curvature is .5 radian per foot.

It also follows from (1) that in different circles, arcs of the same length have a total curvature inversely proportional to their radii.

Thus on a circumference of 1 meter radius, an arc of 5 decimeters has a total curvature of .5 radian, and an average curvature of .1 radian per decimeter; but on a circumference of half a meter radius, the same length of arc has a total curvature of 1 radian and an average curvature of .2 radian per decimeter.

97. Curvature of osculating circle. A curve and its osculating circle at P have the same measure of curvature at that point.

For, let κ, κ' be their respective measures of curvature at the point of contact (x, y). Then from Art. 95,

$$\kappa = \frac{\dfrac{d^2y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}.$$

But this is the reciprocal of the expression for the radius of curvature (Eq. (6), p. 173); hence

$$\kappa = \frac{1}{R}.$$

That is: *the measure of curvature κ at a point P is the reciprocal of the radius of curvature R for that point.* Since a curve and its osculating circle have the same radius of curvature (Art. 90) at their point of contact, it follows from this result that the measure of curvature is also the same for both; $\kappa = \kappa'$.

It is on account of this property that the osculating circle is called the circle of curvature. This is sometimes used as the defining property of the circle of curvature. The radius of curvature at P would then be defined as the radius of the circle whose measure of curvature is the same as that of the given curve at the point P. Its value, as found from Art. 95 and Art. 96, accords with that given in Art. 91.

EXERCISES

1. Find the radius of curvature of the curve $y^2 = 4ax$ at the origin.

2. Find the radius of curvature of the curve $y^3 + x^3 + a(x^2 + y^2) = a^2y$ at the origin.

3. Find the radius of curvature of the curve $a^3y = bx^3 + cx^2y$ at the origin.

Find the center and the radius of curvature for each of the following curves at the point (x, y) and their numerical values at the special point indicated. Find where the curvature is greatest and least on each curve.

4. Rectangular hyperbola $xy = m^2$ at (m, m).

5. Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a, 0)$.

6. General parabola $a^{n-1}y = x^n$ at (a, a).

7. Parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $(a, 0)$.

8. Hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at the point at which $x = y$.

9. Cissoid $y^2 = \frac{x^3}{2a - x}$ at $x = a$.

10. Catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ at $x = 0$.

11. In what points of the curve $y = x^3$ is the curvature greatest?

98. Direct derivation of the expressions for κ and R in polar coördinates. Using the notation of Art. 58, we have

$$\phi = \theta + \psi,$$

hence
$$\kappa = \frac{d\phi}{ds} = \frac{\frac{d\phi}{d\theta}}{\frac{ds}{d\theta}} = \frac{\left(1 + \frac{d\psi}{d\theta}\right)}{\frac{ds}{d\theta}} \tag{1}$$

$$= \frac{\left(1 + \frac{d\psi}{d\theta}\right)}{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{1}{2}}}. \qquad \text{[Art. 44.}$$

But
$$\tan\psi = \rho\frac{d\theta}{d\rho}, \quad \psi = \tan^{-1}\left[\frac{\rho}{\frac{d\rho}{d\theta}}\right],$$

therefore, by differentiating as to θ and reducing, we obtain

$$\frac{d\psi}{d\theta} = \frac{\left(\frac{d\rho}{d\theta}\right)^2 - \rho\frac{d^2\rho}{d\theta^2}}{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2},$$

which, substituted in (1), gives

$$\kappa = \frac{\rho^2 - \rho\dfrac{d^2\rho}{d\theta^2} + 2\left(\dfrac{d\rho}{d\theta}\right)^2}{\left[\rho^2 + \left(\dfrac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

Since $\kappa = \dfrac{1}{R}$, it follows that

$$R = \frac{\left[\rho^2 + \left(\dfrac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2 - \rho\dfrac{d^2\rho}{d\theta^2} + 2\left(\dfrac{d\rho}{d\theta}\right)^2}.$$

This result should be compared with that of Art. 72.

When $u = \dfrac{1}{\rho}$ is taken as dependent variable, the expression for κ assumes the simpler form

$$\kappa = \frac{u^3\left(u + \dfrac{d^2u}{d\theta^2}\right)}{\left[u^2 + \left(\dfrac{du}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

Since at a point of inflexion κ vanishes and changes sign, hence the condition for a point of inflexion, expressed in polar coördinates, is that $u + \dfrac{d^2u}{d\theta^2}$ shall vanish and change sign.

EXERCISES

Find the radius of curvature for each of the following curves:

1. $\rho = a^\theta$.
2. $\rho^2 = a^2 \cos 2\,\theta$.
3. $\rho = 2\,a \cos\theta - a$.
4. $\rho \cos^2 \frac{1}{2}\,\theta = a$.
5. $\rho^2 \cos 2\,\theta = a^2$.
6. $\rho = 2\,a(1 - \cos\theta)$.
7. $\rho\theta = a$.

EVOLUTES AND INVOLUTES

99. Definition of an evolute. When the point P moves along the given curve, the center of curvature C describes another curve which is called the *evolute* of the first.

Let $f(x, y)=0$ be the equation of the given curve. Then the equation of the locus described by the point C is found by eliminating x and y from the three equations

$$f(x, y)=0,$$

$$x-\alpha=\frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}},$$

$$y-\beta=-\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}},$$

and thus obtaining a relation between α, β, the coördinates of the center of curvature.

No general process of elimination can be given; the method to be adopted depends upon the form of the given equation $f(x,y)=0$.

Even when the elimination cannot be performed, the evolute can be traced from point to point by computing successive values of (α, β) corresponding to successive values of (x, y).

Ex. **1.** Find the evolute of the parabola $y^2=4\,px$.

Since $\qquad y=2\,p^{\frac{1}{2}}x^{\frac{1}{2}},\ \frac{dy}{dx}=p^{\frac{1}{2}}x^{-\frac{1}{2}},\ \frac{d^2y}{dx^2}=-\frac{1}{2}p^{\frac{1}{2}}x^{-\frac{3}{2}},$

hence $\qquad x-\alpha=-p^{\frac{1}{2}}x^{-\frac{1}{2}}(1+px^{-1})2\,p^{-\frac{1}{2}}x^{\frac{3}{2}}=-2\,(x+p),$

and $\qquad y-\beta=(1+px^{-1})2\,p^{-\frac{1}{2}}x^{\frac{3}{2}}=2(p^{-\frac{1}{2}}x^{\frac{3}{2}}+p^{\frac{1}{2}}x^{\frac{1}{2}});$

therefore $\qquad \alpha=2\,p+3\,x, \qquad \beta=-2\,p^{-\frac{1}{2}}x^{\frac{3}{2}}.$

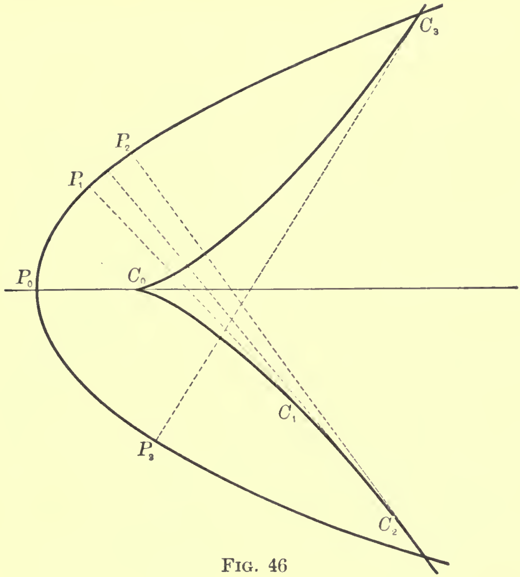

FIG. 46

The result of eliminating x between the last two equations is

$$\tfrac{1}{27}(\alpha - 2p)^3 = \tfrac{1}{4}(p^{\frac{1}{2}}\beta)^2,$$

i.e.,
$$4(\alpha - 2p)^3 = 27\,p\beta^2,$$

which is the equation of the evolute of the parabola, α, β being the current coördinates.

Use the expressions for α and β to compute their values, and to locate the points (α, β) when $x = 0,\ \dfrac{p}{4},\ p$.

Ex. **2.** Find the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

Here
$$\frac{x}{a^2} + \frac{y}{b^2} \cdot \frac{dy}{dx} = 0, \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y},$$

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2} \cdot \frac{y - x\dfrac{dy}{dx}}{y^2} = \frac{-b^2}{a^2y^2}\left(y + \frac{b^2x^2}{a^2y}\right) = \frac{-b^2}{a^4y^3}(a^2y^2 + b^2x^2) = \frac{-b^4}{a^2y^3},$$

whence

$$y-\beta=\frac{(a^4y^2+b^4x^2)y}{a^2b^4}=\left(\frac{a^2y^2}{b^4}+\frac{x^2}{a^2}\right)y=\left(\frac{a^2y^2}{b^4}+1-\frac{y^2}{b^2}\right)y.$$

Therefore
$$-\beta=\frac{a^2-b^2}{b^4}y^3. \tag{2}$$

Similarly,
$$\alpha=\frac{a^2-b^2}{a^4}x^3. \tag{3}$$

On eliminating x, y between (1), (2), (3), the equation of the locus described by (α, β) is

$$(a\alpha)^{\frac{2}{3}}+(b\beta)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}. \qquad \text{(Fig. 51)}$$

Use (2), (3) to locate various values of (α, β), and trace the evolute.

Take
$$a=2b;\ a=\frac{4b}{3}.$$

100. Properties of the evolute. The evolute has two important properties that will now be established.

I. *The normal to the curve is tangent to the evolute.* The relations connecting the coördinates (α, β) of the center of curvature with the coördinates (x, y) of the corresponding point on the curve are, by Art. 91,

$$x-\alpha+(y-\beta)\frac{dy}{dx}=0, \tag{1}$$

$$1+\left(\frac{dy}{dx}\right)^2+(y-\beta)\frac{d^2y}{dx^2}=0. \tag{2}$$

By differentiating (1) as to x, considering α, β, y as functions of x, we obtain

$$1+\left(\frac{dy}{dx}\right)^2+(y-\beta)\frac{d^2y}{dx^2}-\frac{d\alpha}{dx}-\frac{d\beta}{dx}\frac{dy}{dx}=0. \tag{3}$$

Subtracting (3) from (2), we obtain

$$\frac{d\alpha}{dx}+\frac{d\beta}{dx}\frac{dy}{dx}=0, \tag{4}$$

whence
$$\frac{d\beta}{d\alpha}=-\frac{dx}{dy}.$$

But $\frac{d\beta}{d\alpha}$ is the slope of the tangent to the evolute at (α, β), and $-\frac{dx}{dy}$ is the slope of the normal to the given curve at (x, y). Hence these lines have the same slope; but they pass through the same point (α, β), therefore they are coincident.

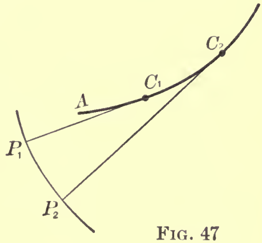

FIG. 47

II. *The difference between two radii of curvature of the given curve, which touch the evolute at the points* C_1, C_2 (Fig. 47), *is equal to the arc* C_1C_2 *of the evolute.*

Since R is the distance between the points (x, y), (α, β),

hence
$$(x-\alpha)^2+(y-\beta)^2=R^2. \tag{5}$$

When the point (x, y) moves along the given curve, the point (α, β) moves along the evolute, and thus α, β, R, y, are all functions of x.

Differentiation of (5) as to x gives

$$(x-\alpha)\left(1-\frac{d\alpha}{dx}\right)+(y-\beta)\left(\frac{dy}{dx}-\frac{d\beta}{dx}\right)=R\frac{dR}{dx}; \tag{6}$$

hence, subtracting (6) from (1), we obtain

$$(x-\alpha)\frac{d\alpha}{dx}+(y-\beta)\frac{d\beta}{dx}=-R\frac{dR}{dx}. \tag{7}$$

Again, from (1) and (4),

$$\frac{\frac{d\alpha}{dx}}{x-\alpha}=\frac{\frac{d\beta}{dx}}{y-\beta}. \tag{8}$$

Hence, each of these fractions is equal to

$$\frac{\sqrt{\left(\dfrac{d\alpha}{dx}\right)^2+\left(\dfrac{d\beta}{dx}\right)^2}}{\sqrt{(x-\alpha)^2+(y-\beta)^2}}=\pm\frac{\dfrac{d\sigma}{dx}}{R}, \tag{9}$$

in which σ is the arc of the evolute. (Compare Art. 41.)

Next, multiplying numerator and denominator of the first member of (8) by $x-\alpha$, and those of the second member by $y-\beta$, and combining new numerators and denominators, we find that each of the fractions in (8) is equal to

$$\frac{(x-\alpha)\dfrac{d\alpha}{dx}+(y-\beta)\dfrac{d\beta}{dx}}{(x-\alpha)^2+(y-\beta)^2},$$

which equals $-\dfrac{R\dfrac{dR}{dx}}{R^2}$ by (7) and (5).

By combining with (9), we obtain

$$\frac{d\sigma}{dx}=\pm\frac{dR}{dx},$$

that is, $$\frac{d}{dx}(\sigma\pm R)=0.$$

Therefore $$\sigma\pm R=\text{constant}, \tag{10}$$

wherein σ is measured from a fixed point A on the evolute.

Now, let C_1, C_2 be the centers of curvature for the points P_1, P_2 on the given curve; let $P_1C_1=R_1$, $P_2C_2=R_2$; and let the arcs AC_1, AC_2 be denoted by σ_1, σ_2. Then

$$\sigma_1\pm R_1=\sigma_2\pm R_2,\text{ by (10)};$$

that is $$\sigma_1-\sigma_2=\pm(R_2-R_1);$$

hence, $$\text{arc } C_1C_2=R_2-R_1. \tag{11}$$

Thus, in Fig. 48,

$$P_1C_1 + C_1C_2 = P_2C_2,$$
$$P_2C_2 + C_2C_3 = P_3C_3, \text{ etc.}$$

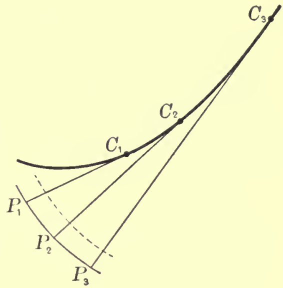

FIG. 48

Hence, if a thread is wrapped around the evolute, and then is unwound, the free end of it can be made to trace out the original curve. From this property the locus of the centers of curvature of a given curve is called the *evolute* of that curve, and the latter is called the *involute* of the former.

When the string is unwound, each point of it describes a different involute; hence, any curve has an infinite number of involutes, but only one evolute.

Any two of these involutes intercept a constant distance on their common normal, and are called *parallel* curves on account of this property.

Ex. Find the length of that part of the evolute of the parabola which lies inside the curve.

From Fig. 46 the required length is twice the difference between the tangents C_3P_3 and P_0C_0, both of which are normals to the parabola.

To find the coördinates of the point P_3, write the equation of the tangent to the evolute at C_3, and find the other point at which it intersects the parabola.

The coördinates of C_3, the point of intersection of the two curves, are $(8p, 4p\sqrt{2})$, and the equation of the tangent at C_3 is

$$2x - \sqrt{2}y - 8p = 0.$$

This tangent intersects the parabola at the point $(2p, -2\sqrt{2}p)$, which is P_3.

The value of the radius of curvature is $\frac{2(x+p)^{\frac{3}{2}}}{\sqrt{p}}$, hence $P_0C_0 = 2p$, $P_3C_3 = 6\sqrt{3}\,p$, hence the arc C_0C_3 is $2p(3\sqrt{3}-1)$, and the required length of the evolute is therefore $4p(3\sqrt{3}-1)$.

EXERCISES

Find the coördinates of the center of curvature for each of the following curves:

1. $x^2 + y^2 = a^2$.

2. $x = a \log \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$.

3. $y^3 = a^2 x$.

4. $y = \frac{c}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

Find the equations of the evolutes of the following curves:

5. $xy = a^2$.

6. $a^2y^2 - b^2x^2 = -a^2b^2$.

7. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

8. Show that the curvature of an ellipse is a minimum at the end of the minor axis, and that the osculating circle at this point has contact of the third order with the curve.

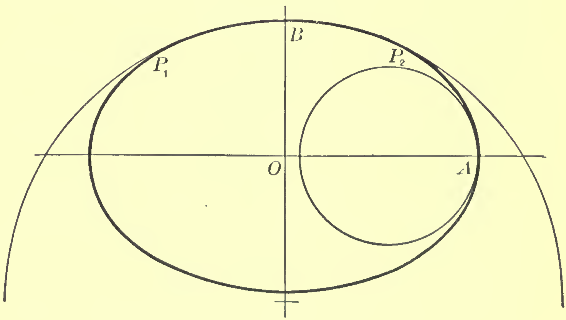

FIG. 49

This circle of curvature must be entirely outside the ellipse (Fig. 49). For, consider two points P_1, P_2, one on each side of B, the end of the minor axis. At these points the curvature is greater than at B, hence these points must be farther from the tangent at B than the circle of curvature, which has everywhere the same curvature as at B.

9. Similarly, show that the curvature at A, the end of the major axis, is a maximum, and that the circle of curvature at A lies entirely within the ellipse (Fig. 49).

10. Show how to sketch the circle of curvature for points between A and B. The circle of curvature for points between A and B has three coincident points in common with the ellipse (Art. 93), hence the circle crosses the curve (Art. 89). Let K, P, L be three points on the arc, such that K is nearest A and L nearest B. The center

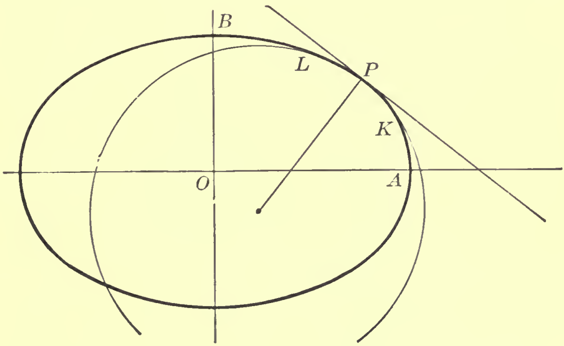

FIG. 50

of curvature for P lies on the normal to P, and on the concave side of the curve. The circle crosses at P, lying outside of the ellipse at K (on the side towards A), and inside the ellipse at L; for the bending of the ellipse increases from B to P and from P to K, while the bending (curvature) of the osculating circle remains constant (Fig. 50).

11. Two centers of curvature lie on every normal. Prove geometrically that the normals to the curve are tangent to the evolute.

12. Show that the entire length of the evolute of the ellipse is $4\left(\frac{a^2}{b} - \frac{b^2}{a}\right)$. [From equation (11) above, take R_1, R_2 as the radii of curvature at the extremities of the major and minor axes.]

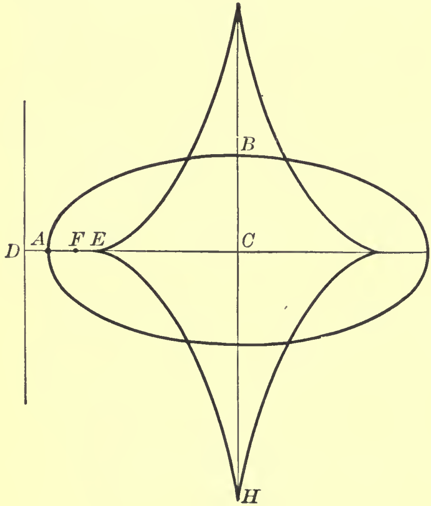

FIG. 51

13. If E is the center of curvature at the vertex A (Fig. 51), prove that $CE = ae^2$, in which e is the eccentricity of the ellipse; and hence that CD, CA, CF, CE form a geometric series whose common ratio is e. Show also that DA, AF, FE form a similar series.

14. If H is the center of curvature for B, show that the point H is without or within the ellipse, according as $a >$ or $< b\sqrt{2}$, or according as $e^2 >$ or $< \frac{1}{2}$. Sketch the evolute when $b = \frac{7a}{8}$.

15. Show by inspection of the figure that four real normals can be drawn to the ellipse from any point within the evolute.

16. Find the parametric equations of the evolute of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

CHAPTER XIII

SINGULAR POINTS

101. Definition of a singular point. If the equation $f(x, y) = 0$ is represented by a curve, the derivative $\frac{dy}{dx}$, when it has a determinate value, measures the slope of the tangent at the point (x, y). There may be certain points on the curve, however, at which the expression for the derivative assumes an illusory or indeterminate form; and, in consequence, the slope of the tangent at such a point cannot be directly determined by the method of Art. 5. Such values of x, y are called *singular values*, and the corresponding points on the curve are called *singular points.*

102. Determination of singular points of algebraic curves. When the equation of the curve is rationalized and cleared of fractions, let it take the form $f(x, y) = 0$.

This gives, by differentiation with regard to x, as in Art. 65,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0,$$

whence

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \tag{1}$$

In order that $\frac{dy}{dx}$ may become illusory, it is therefore necessary to have

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0. \tag{2}$$

Thus, to determine whether a given curve $f(x, y) = 0$ has singular points, put $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ each equal to zero and solve these equations for x and y.

If any pair of values of x and y, so found, satisfy the equation $f(x, y) = 0$, the point determined by them is a singular point on the curve.

To determine the appearance of the curve in the vicinity of a singular point (x_1, y_1), evaluate the indeterminate form

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = \frac{0}{0},$$

by finding the limit approached continuously by the slope of the tangent when $x \doteq x_1$, $y \doteq y_1$.

Hence
$$\frac{dy}{dx} = -\frac{\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial x}\right)}{\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y}\right)}$$

$$= -\frac{\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial x\,\partial y}\dfrac{dy}{dx}}{\dfrac{\partial^2 f}{\partial x\,\partial y} + \dfrac{\partial^2 f}{\partial y^2}\dfrac{dy}{dx}} \qquad \text{[Arts. 64, 85.}$$

at the point (x_1, y_1).

This equation cleared of fractions gives, to determine the slope at (x_1, y_1), the quadratic

$$\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}\left(\frac{dy}{dx}\right) + \frac{\partial^2 f}{\partial x^2} = 0. \tag{3}$$

This quadratic equation has in general two roots. The only exceptions occur when simultaneously, at the point in question,

$$\frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial x\,\partial y} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 0, \tag{4}$$

in which case $\frac{dy}{dx}$ is still indeterminate in form, and must be evaluated as before. The result of the next evaluation is a cubic in $\frac{dy}{dx}$, which gives three values to the slope, unless all the third partial derivatives vanish simultaneously at the singular point.

The geometric interpretation of the two roots of equation (3) will now be given, and similar principles will apply when the quadratic is replaced by an equation of higher degree.

The two roots of (3) are real and distinct, real and coincident, or imaginary, according as

$$H \equiv \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}$$

is positive, zero, or negative. These three cases will be considered separately.

103. Multiple points. First let H be positive. Then at the point (x, y) for which $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, there are two values of the slope, and hence two distinct singular tangents. It follows from this that the curve goes through the point in two directions, or, in other words, two branches of the curve cross at this point. Such a point is called a real double point of the curve, or simply a *node.* The conditions, then, to be satisfied at a node (x_1, y_1) are

$$f(x_1, y_1) = 0, \quad \frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial y_1} = 0,$$

and $$H(x_1, y_1) > 0.$$

Ex. Examine for singular points the curve

$$3\,x^2 - xy - 2\,y^2 + x^3 - 8\,y^3 = 0.$$

Here $$\frac{\partial f}{\partial x} = 6x - y + 3x^2, \quad \frac{\partial f}{\partial y} = -x - 4y - 24y^2.$$

The values $x = 0$, $y = 0$ will satisfy these three equations, hence (0, 0) is a singular point.

Since
$$\frac{\partial^2 f}{\partial x^2} = 6 + 6x = 6 \text{ at } (0, 0),$$
$$\frac{\partial^2 f}{\partial x\, \partial y} = -1,$$
$$\frac{\partial^2 f}{\partial y^2} = -4 - 48y = -4 \text{ at } (0, 0),$$

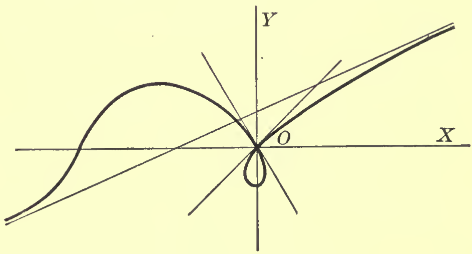

Fig. 52

hence the equation determining the slope is, from (3),

$$-4\left(\frac{dy}{dx}\right)^2 - 2\left(\frac{dy}{dx}\right) + 6 = 0,$$

of which the roots are 1 and $-\frac{3}{2}$. It follows that (0, 0) is a double point at which the tangents have the slopes 1, $-\frac{3}{2}$.

Find the equation of the real asymptote, and the coördinates of the finite point in which it meets the curve.

104. Cusps. Next let $H = 0$. The two tangents are then coincident, and there are two cases to consider. If the curve recedes from the tangent in both directions from the point of tangency, the singular point is called a *tacnode.* Two distinct branches of the curve touch each other at this point. (See Fig. 53.)

If both branches of the curve recede from the tangent in only one direction from the point of tangency, the point is called a *cusp*.

Here again there are two cases to be distinguished. If the branches recede from the point on opposite sides of the double tangent, the cusp is said to be of the first kind; if they recede on the same side, it is called a cusp of the second kind.

The method of investigation will be illustrated by a few examples.

Ex. 1. $$f(x, y) = a^4y^2 - a^2x^4 + x^6 = 0.$$

$$\frac{\partial f}{\partial x} = -4a^2x^3 + 6x^5; \quad \frac{\partial f}{\partial y} = 2a^4y.$$

The point $(0, 0)$ will satisfy $f(x, y) = 0$, $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$; hence it is a singular point. Proceeding to the second derivatives, we obtain

$$\frac{\partial^2 f}{\partial x^2} = -12a^2x^2 + 30x^4 = 0 \text{ at } (0, 0),$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = 0,$$

$$\frac{\partial^2 f}{\partial y^2} = 2a^4.$$

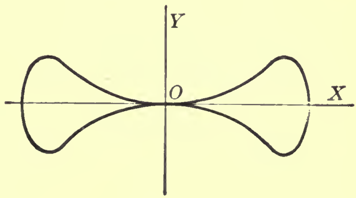

FIG. 53

The two values of $\dfrac{dy}{dx}$ are therefore coincident, and each equal to zero. From the form of the equation, the curve is evidently symmetrical with regard to both axes; hence the point $(0, 0)$ is a tacnode.

No part of the curve can be at a greater distance from the y-axis than $\pm a$, at which points $\dfrac{dy}{dx}$ is infinite. The maximum value of y corresponds to $x = \pm a\sqrt{\frac{2}{3}}$. Between $x = 0$, $x = a\sqrt{\frac{2}{3}}$ there is a point of inflexion (Fig. 53).

Sketch the curves obtained by giving larger and larger values to the parameter a.

Ex. **2.** $f(x, y) = y^2 - x^3 = 0$;

$$\frac{\partial f}{\partial x} = -3x^2, \quad \frac{\partial f}{\partial y} = 2y.$$

Hence the point (0, 0) is a singular point.

$$\text{Further, } \frac{\partial^2 f}{\partial x^2} = -6x = 0 \text{ at } (0, 0);$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = 0; \quad \frac{\partial^2 f}{\partial y^2} = 2.$$

Therefore the two roots of the quadratic equation defining $\frac{dy}{dx}$ are both equal to zero. So far, this case is exactly like the last one, but here no part of the curve lies to the left of the axis y. On the right side, the curve is symmetric with regard to the x-axis. As x increases, y increases; there are no maxima nor minima, and no inflexions (Fig. 54).

Ex. **3.** $f(x, y) = x^4 - 2ax^2y - axy^2 + a^2y^2 = 0.$

The point (0, 0) is a singular point, and the roots of the quadratic defining $\frac{dy}{dx}$ are both equal to zero, hence the origin is a cusp, and the cuspidal tangent is the x-axis.

To show the form of the curve near the cusp, solve the equation for y. Then

$$y = \frac{x^2}{a - x}\left(1 \pm \sqrt{\frac{x}{a}}\right).$$

First suppose that a is positive.

When x is negative, y is imaginary; when $x = 0$, $y = 0$; when x is positive, but less than a, y has two positive values, therefore two branches are above the x-axis. When $x = a$, one branch becomes infinite, having the asymptote $x = a$; the other branch has the ordinate $\frac{1}{2}a$. The origin is therefore a cusp of the second kind (Fig. 55).

Next suppose that a is negative. When x is positive, y is imaginary; when x is negative, y is real. The same reasoning as before shows that there is a cusp of the second kind in the second quarter, with the x-axis as a cuspidal tangent.

Examine the transition case in which $a = 0$.

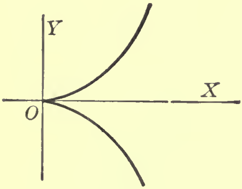

FIG. 54

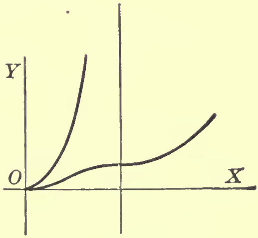

FIG. 55

105. Conjugate points. Lastly, let H be negative. In this case there are no real tangents; hence no points in the immediate vicinity of the given point satisfy the equation of the curve.

Such an isolated point is called a *conjugate point.*

Ex. 1. $f(x, y) = ay^2 - x^3 + bx^2 = 0.$

Here $(0, 0)$ is a singular point of the locus, and at this point we find

$$\frac{dy}{dx} = \pm\sqrt{\frac{-b}{a}},$$

both roots being imaginary if a and b have the same sign.

To show the form of the curve, solve the given equation for y.

Then $$y = \pm x\sqrt{\frac{x-b}{a}},$$

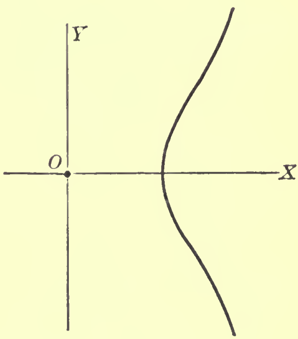

FIG. 56

and hence, if a and b are positive, there are no real points on the curve between $x = 0$ and $x = b$. Thus O is an isolated point (Fig. 56).

Examine the cases in which a or b is negative.

These are the only singularities that algebraic curves can have, although complicated combinations of them may appear. In each of the foregoing examples, the singular point was (0, 0); but for any other point, the same reasoning will apply.

Ex. 2. $f(x, y) = x^2 + 3y^3 - 13y^2 - 4x + 17y - 3 = 0,$

$$\frac{\partial f}{\partial x} = 2x - 4, \quad \frac{\partial f}{\partial y} = 9y^2 - 26y + 17.$$

At the point (2, 1), $f(2, 1) = 0,\ \dfrac{\partial f}{\partial x} = 0,\ \dfrac{\partial f}{\partial y} = 0$; hence (2, 1) is a singular point.

Also $\dfrac{\partial^2 f}{\partial x^2} = 2;\ \dfrac{\partial^2 f}{\partial x\, \partial y} = 0;\ \dfrac{\partial^2 f}{\partial y^2} = 18y - 26, = -8$ at (2, 1).

Hence $\dfrac{dy}{dx} = \pm \dfrac{1}{2}$; and thus the equations of the two tangents at the node (2, 1) are $y - 1 = \frac{1}{2}(x - 2),\ y - 1 = -\frac{1}{2}(x - 2)$.

When H is negative, the singular point is necessarily a conjugate point, but the converse is not always true. A singular point may be a conjugate point when $H = 0$. [Compare Ex. 4 below.]

EXERCISES ON CHAPTER XIII

Examine each of the following curves for multiple points and find the equations of the tangents at each such point; also find the asymptotes and sketch the curve:

1. $a^2x^2 = b^2y^2 + x^2y^2$.

2. $y^2 = \dfrac{x^3}{2a - x}$.

3. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; or, in rational form, $(x^2 + y^2 - a^2)^3 + 27a^2x^2y^2 = 0$.

4. $y^2(x^2 - a^2) = x^4$.

5. $y = a + x + bx^2 \pm cx^{\frac{5}{2}}$; or, in rational form,

$$(y - a - x - bx^2)^2 - c^2x^5 = 0.$$

When a curve has two parallel asymptotes it is said to have a node at infinity in the direction of the parallel asymptotes. Apply to Ex. 6.

6. $(x^2 - y^2)^2 - 4y^2 + y = 0.$

7. $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0.$

8. $y^2 = x(x + a)^2;\ a > 0;\ a < 0.$

9. $x^3 - 3axy + y^3 = 0.$ Find the asymptote and sketch the curve.

10. $y^2 = x^4 + x^5.$

11. Show that the curve $y = x \log x$ has a terminating point at the origin. Find the minimum value of y and sketch the curve.

12. $y = x^2 \log x.$

CHAPTER XIV

ENVELOPES

106. Family of curves. The equation of a curve,

$$f(x, y) = 0,$$

usually involves, besides the variables x and y, certain coefficients that serve to fix the size, shape, and position of the curve. The coefficients are called constants with reference to the variables x and y, but it has been seen in previous chapters that they may take different values in different problems, while the form of the equation is preserved. Let α be one of these "constants." Then if α is given a series of numerical values, and if the locus of the equation, corresponding to each special value of α is traced, a series of curves is obtained, all having the same general character, but differing somewhat from each other in size, shape, or position. A system of curves so obtained is called a *family* of curves.

For example, if h, k are fixed, and p is arbitrary, the equation $(y-k)^2 = 2p(x-h)$ represents a family of parabolas, each curve of which has the same vertex (h, k), and the same axis $y = k$, but a different latus rectum. Again, if k is the arbitrary constant, this equation represents a family of parabolas having parallel axes, the same latus rectum, and having their vertices on the same line $x = h$.

The presence of an arbitrary constant α in the equation of a curve is indicated in functional notation by writing the equation in the form $f(x, y, \alpha) = 0$. The quantity α, which is constant for the same curve but different for different curves, is called the *parameter* of the family. The equations of two neighboring curves are then written

$$f(x, y, \alpha) = 0, \quad f(x, y, \alpha + h) = 0,$$

in which h is a small increment of α. These curves can be brought as near to coincidence as desired by diminishing h.

107. Envelope of a family of curves. A point of intersection of two neighboring curves of the family tends toward a limiting position as the curves approach coincidence. The locus of such limiting points of intersection is called the *envelope* of the family.

Let $$f(x, y, \alpha) = 0, \quad f(x, y, \alpha + h) = 0 \tag{1}$$

be two curves of the family. By the theorem of mean value (Art. 39)

$$f(x, y, \alpha + h) = f(x, y, \alpha) + h\frac{\partial f}{\partial \alpha}(x, y, \alpha + \theta h), \tag{2}$$

which, on account of equation (1), reduces to

$$\frac{\partial f}{\partial \alpha}(x, y, \alpha + \theta h) = 0.$$

Hence, it follows that in the limit, when $h \doteq 0$,

$$\frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0$$

is the equation of a curve passing through the limiting points of intersection of the curve $f(x, y, \alpha) = 0$ with its consecutive curve. This determines for any assigned value of α definite limiting points of intersection on the corresponding member of the family. The locus of all such points is then to be obtained by eliminating the parameter α from the equations

$$f(x, y, \alpha) = 0, \quad \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0.$$

The resulting equation in x and y represents the fixed envelope of the family.

108. The envelope touches every curve of the family.

I. *Geometrical proof.* Let A, B, C (Fig. 57) be three consecutive curves of the family; let A, B intersect in P, and B, C intersect in Q. When P, Q approach coincidence, PQ will be the direction of the tangent to the envelope at P; but since P, Q

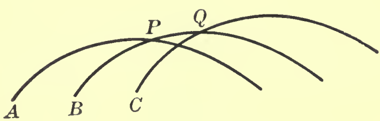

FIG. 57

are two points on B that approach coincidence, hence PQ is also the direction of the tangent to B at P, and accordingly B and the envelope have a common tangent at P. Similarly for every curve of the family.

II. *More rigorous analytical proof.* Let $\dfrac{\partial}{\partial \alpha} f(x, y, \alpha) = 0$ be solved for α, in the form $\alpha = \phi(x, y)$. Then the equation of the envelope is

$$f(x, y, \phi(x, y)) = 0.$$

Equating the total x-derivative to zero, we obtain

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}\right) = 0;$$

but $\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial \alpha} = 0$, hence the slope of the tangent to the envelope at the point (x, y) is given by

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0.$$

But this equation defines the direction of the tangent to the curve $f(x, y, \alpha) = 0$ at the same point, and therefore a limiting point of intersection on any member of the family is a point of contact of this curve with the envelope.

Ex. Find the envelope of the family of lines

$$y = mx + \frac{p}{m}, \tag{1}$$

obtained by varying m.

Differentiate (1) as to m,

$$0 = x - \frac{p}{m^2}. \tag{2}$$

To eliminate m multiply (2) by m and square; square (1) and subtract the first from the second. The envelope is found to be the parabola

$$y^2 = 4\,px.$$

Draw the lines (1) corresponding to

$$m = 1, 2, 3, 4, \infty; \; m = -1, -2, -3, -4.$$

109. Envelope of normals of a given curve. The evolute (Art. 99) was defined as the locus of the centers of curvature. The center of curvature was shown to be the point of intersection of consecutive normals (Art. 92), whence by Art. 107 the envelope of the normals is the evolute.

Ex. Find the envelope of the normals to the parabola $y^2 = 4\,px$.

The equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{-y_1}{2\,p}(x - x_1),$$

or, eliminating x_1 by means of the equation $y_1^2 = 4\,px_1$, we obtain

$$y - y_1 = \frac{y_1^3}{8\,p^2} - \frac{xy_1}{2\,p}. \tag{1}$$

The envelope of this line, when y_1 takes all values, is required. Differentiate as to y_1,

$$-1 = \frac{3\,y_1^2}{8\,p^2} - \frac{x}{2\,p},$$

$$y_1^2 = \frac{4\,p}{3}(x - 2\,p).$$

On substituting this value for y_1 in (1), the result,

$$27\,py^2 = 4(x - 2\,p)^3,$$

is the equation of the required evolute. Show that this semi-cubical parabola has a cusp at $(2\,p, 0)$. Trace the curve.

110. Two parameters, one equation of condition. In many cases a family of curves may have two parameters which are connected by an equation. For instance, the equation of the normal to a given curve contains two parameters x_1, y_1 which are connected by the equation of the curve. In such cases one parameter may be eliminated by means of the given relation, and the other treated as before.

When the elimination is difficult to perform, both equations may be differentiated as to one of the parameters, α, regarding the other parameter β as a function of α. This gives four equations from which α, β, and $\frac{d\beta}{d\alpha}$ may be eliminated, the resulting equation being that of the desired envelope.

Ex. 1. Find the envelope of the line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

the sum of its intercepts remaining constant.

The two equations are
$$\frac{x}{a} + \frac{y}{b} = 1,$$
$$a + b = c.$$

Differentiate both equations as to a;
$$\frac{-x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0,$$
$$1 + \frac{db}{da} = 0.$$

Eliminate $\frac{db}{da}$.

Then $\frac{x}{a^2} = \frac{y}{b^2}$, which reduces to
$$\frac{\frac{x}{a}}{a} = \frac{\frac{y}{b}}{b} = \frac{\frac{x}{a} + \frac{y}{b}}{a + b} = \frac{1}{c};\text{ whence } a = \sqrt{cx},\ b = \sqrt{cy}.$$

Therefore
$$\sqrt{x} + \sqrt{y} = \sqrt{c}$$
is the equation of the desired envelope. [Compare Ex. p. 87.]

This equation when rationalized is
$$(x - y)^2 - 2c(x + y) + c^2 = 0.$$

By turning the coördinate axes through 45°, show that this represents a parabola whose axis bisects the angle between the original axes. Show that the curve touches both these axes. Draw different lines of the family, corresponding to $a = 4, b = 4$; $a = 5, b = 3$; $a = 6, b = 2$; $a = 7, b = 1$; $a = 8, b = 0$; etc.

Ex. **2.** Find the envelope of the family of coaxial ellipses having a constant area.

Here
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$
$$ab = k^2.$$

For symmetry, regard a and b as functions of a single parameter t.

Then by differentiation as to t,

$$\frac{x^2}{a^3}\frac{da}{dt}+\frac{y^2}{b^3}\frac{db}{dt}=0,$$

$$b\frac{da}{dt}+a\frac{db}{dt}=0;$$

hence
$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{1}{2},$$

$$a=\pm x\sqrt{2},\ \ b=\pm y\sqrt{2},$$

and the envelope is the pair of rectangular hyperbolas $xy=\pm\frac{1}{2}k^2$.

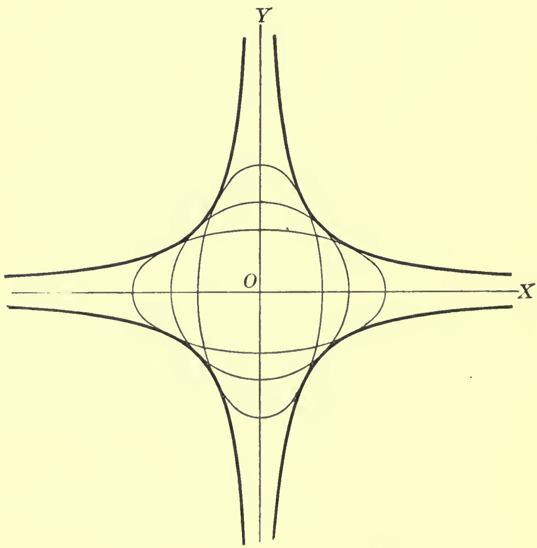

FIG. 58

NOTE. A family of curves may have no envelope; *i.e.*, consecutive curves may not intersect; *e.g.*, the family of concentric circles $x^2+y^2=r^2$, obtained by giving r all possible values.

If every curve of a family has a node, and the node has different positions for different curves of the family, the envelope will be composed of two (or more) curves, one of which is the locus of the node.

Ex. Find the envelope of the system

$$f \equiv (y-\lambda)^2 + x^4 - x^2 = 0,$$

in which λ is a varying parameter.

Here $\dfrac{\partial f}{\partial \lambda} = -2(y-\lambda) = 0$; by combining with $f=0$ to eliminate λ, we obtain $\quad x^2 = 0, \ x-1=0, \ x+1=0.$

From Art. 103 it is seen that the point

$$x = 0, \ y = \lambda$$

is a node on f; moreover, the various curves of the family are obtained by moving any one of them parallel to the y-axis. The lines $x-1=0$, $x+1=0$ form the proper envelope, and $x=0$ is the locus of the node.

EXERCISES ON CHAPTER XIV

Find the envelope of each of the following families of curves; draw to scale various members of the family, and verify that the envelope has been correctly found.

1. The family of straight lines $x\cos\alpha + y\sin\alpha = p$, when α is a parameter.

2. A straight line of fixed length a moving with its extremities in two rectangular axes.

3. Ellipses described with common centers and axes, and having the sum of the semi-axes equal to c.

4. The straight lines having the product of their intercepts on the coördinate axes equal to k^2.

5. The lines $y-\beta = m(x-\alpha) + r\sqrt{1+m^2}$, m being a variable parameter.

6. A circle moving with its center on a parabola whose equation is $y^2 = 4\,ax$, and passing through the vertex of the parabola.

7. A perpendicular to any normal to the parabola $y^2 = 4\,ax$, drawn through the intersection of the normal with the x-axis.

8. The family of circles whose diameters are double ordinates of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

9. The circles which pass through the origin and have their centers on the hyperbola $x^2 - y^2 = c^2$.

10. The family of straight lines $y = 2\,mx + m^4$, m being the variable parameter.

11. The ellipses whose axes coincide, and such that the distance between the extremities of the major and minor axes is constant and equal to k.

12. From a fixed point on the circumference of a circle chords are drawn, and on these as diameters circles are described.

13. With the point (x_1, y_1) on a given ellipse as center, an ellipse is described having its axes equal and parallel to those of the given ellipse. Let (x_1, y_1) describe the given ellipse.

14. Show that if the corner of a rectangular piece of paper is folded down so that the sum of the edges left unfolded is constant, the crease will envelop a parabola.

15. In the "nodal family" $(y - 2\,a)^2 = (x - a)^2 + 8\,x^3 - y^3$, show that the usual process gives for envelope a composite locus, made up of the "node-locus" (a line) and the envelope proper (an ellipse).

16. The family of curves $(y - x^2) + a\,(x - y^2) = 0$.

INTEGRAL CALCULUS

CHAPTER I

GENERAL PRINCIPLES OF INTEGRATION

111. The fundamental problem. The fundamental problem of the Differential Calculus, as explained in the preceding pages, is this:

Given a function $f(x)$ of an independent variable x, to determine its derivative $f'(x)$.

It is now proposed to consider the inverse problem, viz.:

Given any function $f'(x)$, to determine the function $f(x)$ having $f'(x)$ for its derivative.

The solution of this inverse problem is one of the objects of the Integral Calculus.

The given function $f'(x)$ is called the *integrand*, the function $f(x)$ which is to be found is called the *integral*, and the process gone through in order to obtain the unknown function $f(x)$ is called *integration.*

The operation and result of differentiation are symbolized by the formula

$$\frac{d}{dx}f(x) = f'(x), \tag{1}$$

or, written in the notation of differentials,

$$df(x) = f'(x)\,dx. \tag{2}$$

The operation of integration is indicated by prefixing the symbol $\int$ to the function, or differential, whose integral it is required to find. It is called *the integral sign*, or *the sign of integration*. Accordingly, the formula of integration is written thus:

$$f(x) = \int f'(x)\,dx.$$

Following long established usage, the differential, rather than the derivative, of the unknown function $f(x)$ is written under the sign of integration. One of the advantages of so doing is that the variable, with respect to which the integration is performed, is explicitly mentioned. This is, of course, not necessary when only one variable is involved, but it is essential when several variables enter into the integrand, or when a change of variable is made during the process of integration.

112. Integration by inspection. The most obvious aid to integration is a knowledge of the rules and results of differentiation. It frequently happens that the required function $f(x)$ can be determined at once by recollecting the result of some previous differentiation.

For example, suppose it is required to find

$$\int \cos x\,dx.$$

It will be recalled that $\cos x\,dx$ is the differential of $\sin x$, and thus the proposed integration is immediately effected; that is,

$$\int \cos x\,dx = \sin x.$$

Again, suppose it is required to integrate

$$\int x^n\,dx,$$

in which n is any constant (except -1). This problem suggests the formula for differentiating a variable affected by a constant exponent [(6), p. 44]. The formula referred to may be written

$$d\left(\frac{x^{n+1}}{n+1}\right)=x^n\,dx,$$

and hence we conclude, $\displaystyle\int x^n\,dx=\frac{x^{n+1}}{n+1}$.

An exception to this result occurs when n has the value -1. For in that case we deduce from (8), p. 44, the formula of integration

$$\int x^{-1}\,dx=\int\frac{dx}{x}=\log x.$$

The method used in the above illustrations may be designated as *integration by inspection*. This is, in fact, the only practical method available. The object of the various devices suggested in the subsequent pages is to transform the given integrand or to separate it into simpler elements in such a way that the method of inspection can be applied.

113. The fundamental formulas of integration. When the formulas of differentiation, pp. 44–45, are borne in mind, the method of inspection referred to in the preceding article leads at once to the following fundamental integrals. Upon these, sooner or later, every integration must be made to depend.

$$\textbf{I.}\quad \int u^n\,du=\frac{u^{n+1}}{n+1}.$$

$$\textbf{II.}\quad \int\frac{du}{u}=\log u.$$

$$\textbf{III.}\quad \int a^u\,du=\frac{a^u}{\log a}.$$

$$\text{IV.} \quad \int e^u\,du = e^u.$$

$$\text{V.} \quad \int \cos u\,du = \sin u.$$

$$\text{VI.} \quad \int \sin u\,du = -\cos u.$$

$$\text{VII.} \quad \int \sec^2 u\,du = \tan u.$$

$$\text{VIII.} \quad \int \csc^2 u\,du = -\cot u.$$

$$\text{IX.} \quad \int \sec u \tan u\,du = \sec u.$$

$$\text{X.} \quad \int \csc u \cot u\,du = -\csc u.$$

$$\text{XI.} \quad \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u, \text{ or } -\cos^{-1} u.$$

$$\text{XII.} \quad \int \frac{du}{1+u^2} = \tan^{-1} u, \text{ or } -\cot^{-1} u.$$

114. Certain general principles. In applying the above formulas of integration certain principles which follow from the rules of differentiation should be made use of.

(*a*) *The integral of the sum of a finite number of functions is equal to the sum of the integrals of the functions taken separately.*

This follows from Art. 10.

For example,

$$\int \frac{x^2-1}{x}\,dx = \int x\,dx - \int \frac{dx}{x} = \frac{x^2}{2} - \log x.$$

(b) *A constant factor may be removed from one side of the sign of integration to the other.*

For, since $$d(cu) = c\,du,$$

it follows that $$\int c\,du = cu = c\int du.$$

To illustrate, let it be required to integrate

$$\int 5\,x^2\,dx.$$

The numerical factor 5 is first placed outside the sign of integration, after which formula **I** is applied. Accordingly,

$$\int 5\,x^2\,dx = 5\int x^2\,dx = \frac{5\,x^3}{3}.$$

Again, suppose the integral

$$\int \frac{x\,dx}{x^2+1}$$

is to be found. We notice that if the numerator had an additional factor 2, it would be the exact differential of the denominator, and formula **II** would be applicable. All that is required, then, in order to reduce the given integral to a known form, is to multiply inside the sign of integration by 2 and outside by $\frac{1}{2}$. This gives

$$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{2\,x\,dx}{x^2+1} = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2}\log\,(x^2+1).$$

In this connection it must not be forgotten that:

An expression containing the variable of integration cannot be moved from one side of the sign of integration to the other.

(c) *An arbitrary constant may be added to the result of integration.*

For, the derivative of a constant is zero and hence

$$du = d(u + c),$$

from which follows

$$\int du = \int d(u + c) = u + c.$$

This constant is called *the constant of integration.*

From the preceding remark it follows that the result of integration is not unique, but that any number of functions (differing from each other, however, only by an additive constant) can be found, each of which has the same given expression as its derivative. [Compare Art. 10, Cor.]

Thus, any one of the functions $x^2 - 1$, $x^2 + 1$, $x^2 + a^2$, $(x - a)(x + a)$ may serve as a solution of the problem of integrating $\int 2\,x\,dx$.

It often happens that different methods of integration lead to different results. All such differences, however, can occur only in the constant terms.

For example,

$$\int 3\,(x+1)^2\,dx = 3\int (x+1)^2\,d(x+1) = (x+1)^3$$
$$= x^3 + 3\,x^2 + 3\,x + 1.$$

Integration of the terms separately gives

$$\int 3\,x^2\,dx + \int 6\,x\,dx + \int 3\,dx = x^3 + 3\,x^2 + 3\,x,$$

a result that agrees with the preceding except in the constant term.

Again, from formula **XII**,

$$\int \frac{dx}{x^2+1} = \tan^{-1} x, \text{ or } -\cot^{-1} x.$$

It does not follow from this that $\tan^{-1} x$ is equal to $-\cot^{-1} x$. But they can differ at most by an additive constant. In fact, it is known from trigonometry that

$$-\cot^{-1} x = \tan^{-1} x + k\pi + \frac{\pi}{2},$$

in which k is any integer.

In a similar manner the different results in formula **XI** can be explained.

EXERCISES

Integrate the following:

1. $\int \sqrt{x}\, dx.$

[HINT. For the purpose of integration this may be written

$$\int x^{\frac{1}{2}}\, dx.]$$

2. $\int x^a\, dx.$

3. $\int \frac{dx}{\sqrt[3]{x}}.$

4. $\int \frac{mx^{m-1}\, dx}{\sqrt{x}}.$

5. $\int (a^{\frac{1}{3}} - x^{\frac{1}{3}})^3\, dx.$

6. $\int \frac{5x^3 - 3x + 1}{x^4}\, dx.$

7. $\int x(x^2 + a^2)^2\, dx.$

8. $\int (ax + b)^n\, dx.$

9. $\int \frac{dx}{x + a}.$

10. $\int \frac{(a - x)dx}{2ax - x^2}.$

11. $\int \frac{\csc^2 x\, dx}{\cot x}.$

12. $\int \frac{\sin x\, dx}{1 + \cos x}.$

13. $\int \frac{dx}{x \log x} \left[= \int \frac{\frac{dx}{x}}{\log x} \right].$

14. $\int \frac{5x^2\, dx}{x^3 + 1}.$

15. $\int \tan x\, dx \left[= -\int \frac{-\sin x\, dx}{\cos x} \right].$

16. $\int \cot x\, dx.$

17. $\int e^{ax}\, dx.$

18. $\int e^{x^2} x\, dx.$

19. $\int (a + b)^{m+nx}\, dx.$

20. $\int \cos 2x\, dx.$

21. $\int \sin nx\, dx.$

22. $\int \cos^2 x\,dx \left[= \int \frac{1+\cos 2x}{2}\,dx\right]$.

23. $\int \sin^2 x\,dx$. **24.** $\int \sin(m+n)x\,dx$. **25.** $\int x \sin x^2\,dx$.

26. $\int \cos^3 x\,dx \left[= \int (1-\sin^2 x)\cos x\,dx\right]$. **27.** $\int \sin^3 x\,dx$.

28. $\int \tan^2 x\,dx \left[= \int (\sec^2 x - 1)\,dx\right]$. **29.** $\int \tan^2 x \sec^2 x\,dx$.

30. $\int \csc^2 (ax+b)\,dx$. **31.** $\int \sqrt{\cot x}\cdot \csc^2 x\,dx$.

32. $\int \frac{dx}{\sin x \cos x} \left[= \int \frac{\sec^2 x\,dx}{\tan x}\right]$.

33. $\int \sec^3 x \tan x\,dx$.

34. $\int \frac{\tan x\,dx}{\sec x}$. **35.** $\int \frac{dx}{\sqrt{a^2-x^2}}$.

[HINT. Divide numerator and denominator by a and then write in the form

$$\int \frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.\Bigg]$$

36. $\int \frac{dx}{\sqrt{1-4x^2}}$. **37.** $\int \frac{du}{a^2+u^2}$. **38.** $\int \frac{dx}{a^2x^2+b^2}$.

39. $\int \frac{dx}{x^2-4x+5} \left[= \int \frac{d(x-2)}{(x-2)^2+1}\right]$.

115. Integration by parts. If u and v are functions of x, the rule for differentiating a product gives

$$d(uv) = v\,du + u\,dv,$$

whence, by integrating and transposing terms, we have

$$\int u\,dv = uv - \int v\,du.$$

This formula affords a most valuable method of integration, known as *integration by parts.* By its use a given integral is made to depend on another integral, which in many cases is of a simpler form and more readily integrable than the original one.

Ex. 1. $$\int \log x\,dx.$$

Assume $$u = \log x,\quad dv = dx.$$

Then $$du = \frac{dx}{x},\quad v = x.$$

By substituting in the formula for integration by parts, we obtain,

$$\begin{aligned}\int \log x\,dx &= x \log x - \int dx\\ &= x\log x - x = x(\log x - 1)\\ &= x(\log x - \log e) = x \log \frac{x}{e}.\end{aligned}$$

Ex. 2. $$\int xe^x\,dx.$$

Assume $$u = x,\quad dv = e^x\,dx.$$

Then $$du = dx,\quad v = e^x,$$

and $$\int xe^x\,dx = xe^x - \int e^x\,dx = e^x(x-1).$$

Suppose that a different choice had been made for u and dv in the present problem, say $$u = e^x,\quad dv = x\,dx.$$

From this would follow

$$du = e^x\,dx,\quad v = \frac{x^2}{2},$$

and $$\int xe^x\,dx = \tfrac{1}{2}x^2e^x - \int \frac{x^2}{2}e^x\,dx.$$

It will be observed that the new integral $\int \frac{x^2}{2}e^x\,dx$ is less simple in form than the original one; hence the present choice of u and dv is not a fortunate one.

No general rule can be laid down for the selection of u and dv. Several trials may be necessary before a suitable one can be found.

It is to be remarked, however, that dv should be so chosen that its integral may be as simple as possible, while u should be so chosen that in differentiating it a material simplification is brought about. Thus in Ex. 1, by taking $u = \log x$, the transcendental function is made to disappear by differentiation. In Ex. 2, the presence of either x or e^x prevents direct integration. The first factor x can be removed by differentiation, and thus the choice $u = x$ is naturally suggested.

Ex. **3.**
$$\int x^2 a^x\, dx.$$

From the preceding remark it is evident that the only choice which will simplify the integral is

$$u = x^2, \qquad dv = a^x\, dx.$$

Hence
$$du = 2\, x\, dx, \quad v = \frac{a^x}{\log a},$$

and
$$\int x^2 a^x\, dx = \frac{x^2 a^x}{\log a} - \frac{2}{\log a}\int x a^x\, dx.$$

Apply the same method to the new integral, assuming

$$u = x, \quad dv = a^x\, dx,$$

whence
$$du = dx, \quad v = \frac{a^x}{\log a},$$

and
$$\int x a^x\, dx = \frac{x a^x}{\log a} - \frac{1}{\log a}\int a^x\, dx$$

$$= \frac{x a^x}{\log a} - \frac{a^x}{(\log a)^2}.$$

By substituting in the preceding formula, we have

$$\int x^2 a^x\, dx = \frac{a^x}{\log a}\left[x^2 - \frac{2\, x}{\log a} + \frac{2}{(\log a)^2}\right].$$

EXERCISES

1. $\int \sin^{-1} x\,dx.$
2. $\int e^x \tan^{-1}(e^x)\,dx.$
3. $\int x^2 \cos x\,dx.$
4. $\int x^n \log x\,dx.$
5. $\int x^2 \tan^{-1} x\,dx.$
6. $\int \sec x \tan x \log \cos x\,dx.$
7. $\int x \cot^{-1} x\,dx.$
8. $\int x \sin 3x\,dx.$
9. $\int e^x \cos x\,dx.$
10. $\int e^x \sin x\,dx.$
11. $\int \cos x \cos 2x\,dx.$
12. $\int x \sec^2 x\,dx.$

116. Integration by substitution. It is often necessary to simplify a given differential $f'(x)dx$ by the introduction of a new variable before integration can be effected. Except for certain special classes of differentials (see, for example, Arts. 127–129) no general rule can be laid down for the guidance of the student in the use of this method, but some aid may be derived from the hints contained in the problems which follow.

Ex. 1. $\int \frac{x\,dx}{\sqrt{a^2 - x^2}}.$

Introduce a new variable z by means of the substitution $a^2 - x^2 = z$. Differentiate and divide by -2, whence $x\,dx = -\frac{dz}{2}$. Accordingly,

$$\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\frac{1}{2}\int \frac{dz}{\sqrt{z}} = -\frac{1}{2}\int z^{-\frac{1}{2}}\,dz = -z^{\frac{1}{2}} = -\sqrt{a^2 - x^2}.$$

The details required in carrying out this substitution are so simple that they can be omitted and the solution of the problem will then take the following form:

$$\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = \int (a^2 - x^2)^{-\frac{1}{2}} x\,dx = -\frac{1}{2}\int (a^2 - x^2)^{-\frac{1}{2}}(-2x\,dx)$$
$$= -(a^2 - x^2)^{\frac{1}{2}}.$$

In this series of steps the last integral is obtained by multiplying inside the sign of integration by -2 and outside by $-\frac{1}{2}$, the object being to make the second factor the differential of $a^2 - x^2$. Thinking of the latter as a new variable, the integrand contains this variable affected by an exponent $(-\frac{1}{2})$ and multiplied by the differential of the variable, in which case formula **I** can be applied.

Ex. **2.** $\int \frac{\log x}{x}\,dx.$

Assume $$\log x = z.$$

Then $$\frac{dx}{x} = dz,$$

and $$\int \frac{\log x}{x}\,dx = \int z\,dz = \frac{z^2}{2} = \frac{(\log x)^2}{2}.$$

Here again it is not necessary to write out the details of the substitution, as it is easy to think of $\log x$ as a new independent variable and to perform the integration with respect to it. It is then readily seen that the expression to be integrated consists of the variable $\log x$ multiplied by its differential $\frac{dx}{x}$, and that the integration is accordingly reduced to an immediate application of the first formula of integration. Thus

$$\int \log x \cdot d(\log x) = \frac{(\log x)^2}{2}.$$

Ex. **3.** $\int e^{\tan^{-1}x} \frac{dx}{1+x^2}.$

Think of $\tan^{-1} x$ as a new variable and apply formula **IV**. This gives

$$\int e^{\tan^{-1}x} \frac{dx}{1+x^2} = \int e^{\tan^{-1}x}\, d(\tan^{-1} x) = e^{\tan^{-1}x}.$$

Ex. **4.** $\int \frac{\sin^{-1} x\,dx}{\sqrt{1-x^2}}.$

Regard $\sin^{-1} x$ as a new variable and $\frac{dx}{\sqrt{1-x^2}}$ as the differential of that variable. Apply formula **I**.

Ex. **5.** $\int (x^2 + 2x + 3)(x + 1)dx.$

Multiply and divide by 2. The integral then takes the form

$$\frac{1}{2}\int (x^2 + 2x + 3) \cdot (2x + 2)dx.$$

Observing that $(2x + 2)dx$ is the differential of $x^2 + 2x + 3$, and using the latter expression as a new variable, we see that formula **I** is directly applicable, leading to the result

$$\tfrac{1}{4}(x^2 + 2x + 3)^2.$$

Ex. **6.** $\int \log \cos (x^2 + 1) \sin(x^2 + 1) \cdot x\,dx.$

Make the substitution $\quad x^2 + 1 = z.$

The given integral takes the form

$$\frac{1}{2}\int \log \cos z \sin z\,dz.$$

Make a second change of variable,

$$\cos z = y.$$

Then $\quad \sin z\,dz = -dy.$

The transformed integral is

$$-\frac{1}{2}\int \log y\,dy,$$

to which the result of Ex. 1, p. 217, can be at once applied.

It will be observed that two substitutions which naturally suggest themselves from the form of the integrand are made in succession. The two together are obviously equivalent to the one transformation,

$$\cos(x^2 + 1) = y.$$

Ex. **7.** $\int \frac{du}{\sqrt{a^2 - u^2}}.$ Ex. **8.** $\int \frac{du}{u^2 + a^2}.$

[Hint. Substitute $u = az$.]

Ex. **9.** $\int \frac{dx}{x\sqrt{x^2 - a^2}}$.

$\left[\text{Hint.} \quad \text{Substitute } x = \frac{1}{z}.\right]$

Ex. **10.** $\int \frac{dx}{\sqrt{2\,ax - x^2}}$.

[Hint. Substitute $x = z + a$.]

Ex. **11.** $\int \csc u\, du$.

Multiply and divide the integrand by $\csc u - \cot u$. It will then be seen that the integral has the form $\int \frac{dz}{z}$.

Another method would be to use the trigonometric formula

$$\sin u = 2 \sin \frac{u}{2} \cos \frac{u}{2},$$

whence $$\int \csc u\, du = \int \frac{du}{2 \sin \frac{u}{2} \cos \frac{u}{2}} = \int \frac{\sec^2 \frac{u}{2}\, d\left(\frac{u}{2}\right)}{\tan \frac{u}{2}} = \int \frac{dt}{t},$$

in which $t = \tan \frac{u}{2}$.

Ex. **12.** $\int \sec u\, du$.

Put $u = z - \frac{\pi}{2}$ and use Ex. 11.

Solve the problem also by means of substitutions similar to those used in the preceding example.

Ex. **13.** $\int x^2 \sqrt{a^3 - x^3}\, dx$.

Ex. **14.** $\int \frac{x^2\, dx}{(x - 1)^3}$.

Ex. **15.** $\int \frac{\cos x\, dx}{\sin^3 x}$.

Ex. **16.** $\int \frac{dx}{\cos^2 x + 2 \sin^2 x}$.

Put $\tan x = z$.

Ex. **17.** Prove that $\int \frac{x^m\, dx}{(a + bx)^n}$ can be integrated by a substitution, when m is a positive integer.

117. Additional standard forms. The integrals in Exs. 7, 8, 11, 12 of the preceding article, and in Exs. 15, 16 of Art. 114, are of such frequent occurrence that it is desirable to collect the results of integration into an additional list of standard forms. Two other very useful formulas are also included, the derivation of which we now give.

Integration of $$\int \frac{du}{\sqrt{u^2 + a}}.$$

Make the substitution

$$u + \sqrt{u^2 + a} = z.$$

From this equation, we obtain, by differentiation,

$$\left(1 + \frac{u}{\sqrt{u^2 + a}}\right) du = dz;$$

that is, $$(\sqrt{u^2 + a} + u)\frac{du}{\sqrt{u^2 + a}} = dz,$$

whence, $$\frac{du}{\sqrt{u^2 + a}} = \frac{dz}{\sqrt{u^2 + a} + u} = \frac{dz}{z}.$$

This gives, on integrating,

$$\int \frac{du}{\sqrt{u^2 + a}} = \int \frac{dz}{z} = \log z$$

$$= \log(u + \sqrt{u^2 + a}).$$

Integration of $$\int \frac{du}{u^2 - a^2}.$$

The fraction $\frac{1}{u^2 - a^2}$ may be written as the sum of two simpler fractions,

$$\frac{1}{u^2 - a^2} = \frac{1}{2\,a}\left[\frac{1}{u - a} - \frac{1}{u + a}\right],$$

whose denominators are the factors of $u^2 - a^2$. Hence,

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \int \left[\frac{du}{u-a} - \frac{du}{u+a} \right]$$

$$= \frac{1}{2a} \left[\log(u-a) - \log(u+a) \right] = \frac{1}{2a} \log \frac{u-a}{u+a}.$$

XIII. $\displaystyle \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a}.$

XIV. $\displaystyle \int \frac{du}{\sqrt{u^2 + a}} = \log(u + \sqrt{u^2 + a}).$

XV. $\displaystyle \int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a}.$

XVI. $\displaystyle \int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \frac{u-a}{u+a}.$

XVII. $\displaystyle \int \tan u \, du = -\log \cos u = \log \sec u.$

XVIII. $\displaystyle \int \cot u \, du = \log \sin u.$

XIX. $\displaystyle \int \sec u \, du = \log(\sec u + \tan u) = \log \tan \left(\frac{u}{2} + \frac{\pi}{4} \right).$

XX. $\displaystyle \int \csc u \, du = \log(\csc u - \cot u) = \log \tan \frac{u}{2}.$

118. Integrals of the forms

$$\int \frac{(Ax + B)dx}{ax^2 + bx + c} \quad \text{and} \quad \int \frac{(Ax + B)dx}{\sqrt{ax^2 + bx + c}}.$$

Such integrals occur so frequently that they deserve special mention. The integration is facilitated by the substitution of a new variable t which reduces the affected quadratic $ax^2 + bx + c$ to a pure quadratic of the form $mt^2 + n$. The mode of procedure will be readily understood from the following illustrative problems.

Ex. 1. $$\int \frac{x\,dx}{2x^2+2x+3}.$$

The first step is to complete the square of the x terms in the denominator. After the factor 2 has been placed outside the integral sign, the quadratic expression may be written

$$(x^2+x+\tfrac{1}{4})+(\tfrac{3}{2}-\tfrac{1}{4})=(x+\tfrac{1}{2})^2+\tfrac{5}{4}.$$

Now substitute a new variable t in place of $x+\frac{1}{2}$. Since $x=t-\frac{1}{2}$ and $dx=dt$, we obtain for the new form of the given integral

$$\begin{aligned}\frac{1}{2}\int\frac{(t-\frac{1}{2})\,dt}{t^2+\frac{5}{4}}&=\frac{1}{4}\int\frac{2t\,dt}{t^2+\frac{5}{4}}-\frac{1}{4}\int\frac{dt}{t^2+\frac{5}{4}}\\&=\frac{1}{4}\log\left(t^2+\frac{5}{4}\right)-\frac{1}{2\sqrt{5}}\tan^{-1}\frac{2t}{\sqrt{5}}\\&=\frac{1}{4}\log\left(x^2+x+\frac{3}{2}\right)-\frac{1}{2\sqrt{5}}\tan^{-1}\frac{2x+1}{\sqrt{5}}.\end{aligned}$$

Ex. 2. $$\int\frac{(2x-1)dx}{\sqrt{1+2x-3x^2}}.$$

Divide out $\sqrt{3}$ from the denominator; since the coefficient of x^2 is negative, put the x terms in parentheses preceded by the negative sign and complete the square. The integral then becomes

$$\frac{1}{\sqrt{3}}\int\frac{(2x-1)\,dx}{\sqrt{\frac{4}{9}-(x-\frac{1}{3})^2}}.$$

Now make the substitution $x-\frac{1}{3}=t$. Since $dx=dt$, the integral reduces to

$$\begin{aligned}\frac{1}{\sqrt{3}}\int\frac{(2t-\frac{1}{3})dt}{\sqrt{\frac{4}{9}-t^2}}&=-\frac{1}{\sqrt{3}}\int\left(\frac{4}{9}-t^2\right)^{-\frac{1}{2}}\left(-2t\,dt\right)-\frac{1}{3\sqrt{3}}\int\frac{dt}{\sqrt{\frac{4}{9}-t^2}}\\&=-\frac{2}{\sqrt{3}}\left(\frac{4}{9}-t^2\right)^{\frac{1}{2}}-\frac{1}{3\sqrt{3}}\sin^{-1}\left(\frac{3t}{2}\right)\\&=-\frac{2}{\sqrt{3}}\sqrt{\frac{1}{3}+\frac{2}{3}x-x^2}-\frac{1}{3\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{2}\right)\\&=-\frac{2}{3}\sqrt{1+2x-3x^2}-\frac{1}{3\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{2}\right).\end{aligned}$$

It is seen from the two preceding examples that the metl here used contains two essential steps:

(1) *Completing the square of the x terms in $ax^2 + bx + c$;*

(2) *Substituting a new variable for the part in parentheses.*

If the numerator of the new integral contains two terms, separate into two integrals and integrate each one separately.

EXERCISES

1. $\int \frac{dx}{2x^2 + 4x + 1}$.

2. $\int \frac{dx}{3x^2 - 2x + 5}$.

3. $\int \frac{dx}{8 + 4x - 4x^2}$.

4. $\int \frac{dx}{\sqrt{30x - 9x^2 - 24}}$.

5. $\int \frac{x\,dx}{\sqrt{x^2 + 2x + 2}}$.

6. $\int \frac{x\,dx}{\sqrt{1 + 2x - x^2}}$.

7. $\int \frac{(4x + 5)dx}{\sqrt{8 - 4x - 4x^2}}$.

8. $\int \frac{(2x - 3)dx}{\sqrt{3x^2 + x - 2}}$.

9. $\int \sqrt{\frac{1 - x}{1 + 2x}}\,dx$.

[Rationalize the numerator.]

10. $\int \frac{(3x + 2)dx}{x^2 - 6x + 5}$.

11. $\int \sqrt{\frac{a - x}{x}}\,dx$.

12. $\int \frac{(2x + 1)dx}{\sqrt{-2x^2 - 3x - 1}}$.

13. $\int \frac{(x - 3)dx}{\sqrt{-3x^2 - 2x + 1}}$.

119. Integrals of the forms

$$\int \frac{dx}{(Ax + B)\sqrt{ax^2 + bx + c}} \text{ and } \int \frac{dx}{(Ax + B)^2\sqrt{ax^2 + bx + c}}.$$

Integrals of these types can be reduced to forms given in the preceding article by means of the reciprocal substitution

$$Ax + B = \frac{1}{t},\ dx = -\frac{dt}{At^2}.$$

EXERCISES

1. $\int \frac{dx}{x\sqrt{x^2 + a^2}}$.

2. $\int \frac{dx}{x\sqrt{a^2 - x^2}}$.

3. $\int \frac{dx}{x\sqrt{5\,x^2 - 4\,x + 1}}$.

4. $\int \frac{dx}{(x + 1)\sqrt{x^2 + 2\,x + 3}}$.

5. $\int \frac{dx}{(x + 1)\sqrt{x^2 + x + 1}}$.

6. $\int \frac{dx}{(1 - x)\sqrt{2\,x^2 - 4\,x + 1}}$.

7. $\int \frac{dx}{(x + 2)\sqrt{- x^2 - 10\,x - 7}}$

8. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}}$.

9. $\int \frac{dx}{x^2\sqrt{x^2 - a^2}}$.

10. $\int \frac{dx}{x^2\sqrt{x^2 + a^2}}$.

11. $\int \frac{2\,dx}{(2\,x - 1)\sqrt{4\,x^2 - 3}}$.

EXERCISES ON CHAPTER I

1. $\int e^{e^x} e^x\,dx$.

2. $\int \frac{5\,x^3\,dx}{4 + x^8}$.

3. $\int \frac{(2 + 3\,x^2)\,dx}{6\,x^3 + 12\,x + 5}$.

4. $\int \frac{1 + x}{\sqrt{x}}\,dx$.

5. $\int \frac{dx}{\sqrt[3]{3 - 2\,x}}$.

6. $\int \frac{dx}{(a - bx)^8}$.

7. $\int x(a^2 - x^2)^{\frac{1}{3}}\,dx$.

8. $\int \frac{x\,dx}{(x^2 + 1)^{\frac{3}{2}}}$.

9. $\int \frac{dx}{\sqrt{x + 1} + \sqrt{x - 1}}$.

10. $\int \cos 8\,x\,dx$.

11. $\int \sec 3\,x\,dx$.

12. $\int e^x \sin e^x\,dx$.

13. $\displaystyle\int \frac{\sin x\, dx}{a \cos x + b}$.

14. $\displaystyle\int \frac{dx}{\sqrt{1 - e^{2x}}}$.

[Put $e^{-x} = t$.]

15. $\displaystyle\int \frac{x\, dx}{\sqrt{1 - x^4}}$.

16. $\displaystyle\int \frac{dx}{\sqrt{4x^4 + 8x^2}}$.

17. $\displaystyle\int \frac{dx}{e^x - e^{-x}}$.

18. $\displaystyle\int x^4 \tan^{-1} x\, dx$.

19. $\displaystyle\int \frac{x^2\, dx}{a^x}$.

20. $\displaystyle\int \frac{d\theta}{1 + \sin\theta}\left[= \int \frac{1 - \sin\theta}{\cos^2\theta} d\theta\right]$.

21. $\displaystyle\int \frac{d\theta}{1 - \cos\theta}$.

22. $\displaystyle\int \frac{\tan\theta\, d\theta}{a + b\tan^2\theta}$.

23. $\displaystyle\int \frac{dx}{6x\sqrt{1 - \log x}}$.

24. $\displaystyle\int \frac{e^x\, dx}{e^x + e^{-x}}$.

25. $\displaystyle\int \frac{\cos\theta\, d\theta}{\sqrt{1 + \cos^2\theta - \sin\theta}}$.

26. $\displaystyle\int \frac{dx}{x(\log x)^2 + x}$.

27. $\displaystyle\int \left(\frac{\sec x}{a - b\tan x}\right)^2 dx$.

28. $\displaystyle\int \frac{(x - a)\, dx}{\sqrt{a^4 - a^2(x - a)^2 - (x - a)^4}}$.

29. $\displaystyle\int \frac{dx}{x^2\sqrt{3x^2 + 2x + 1}}$.

30. $\displaystyle\int \log(x + \sqrt{x^2 - a^2})\frac{dx}{x^3}$.

31. $\displaystyle\int \sin x \log \tan x\, dx$.

32. $\displaystyle\int \frac{dx}{1 + \cot x}\left[= \int \frac{\sin x\, dx}{\sin x + \cos x}\right.$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{\sin x + \cos x} dx$$

$$\left.= \frac{1}{2}\int \left(1 - \frac{\cos x - \sin x}{\sin x + \cos x}\right) dx\right].$$

[Another method would be to multiply numerator and denominator by $\sin x(\cos x - \sin x)$ and express in terms of the double angle.]

CHAPTER II

REDUCTION FORMULAS

120. In Arts. 118, 119 the integration of certain simple expressions containing an irrationality of the form $\sqrt{ax^2+bx+c}$ was explained. As was shown in Art. 118, the radical can be reduced to the form $\sqrt{\pm x^2 \pm a^2}$ by a change of variable. It remains to show how the integration can be performed in in such cases as, for example,

$$\int x^n\sqrt{\pm x^2 \pm a^2}dx, \qquad \int \frac{x^n dx}{\sqrt{\pm x^2 \pm a^2}},$$

n being any integer.

For this purpose it is convenient to consider a more general type of integral of which the preceding are special cases, viz.,

$$\int x^m(a + bx^n)^p dx, \tag{1}$$

in which m, n, p are any numbers whatever, integral or fractional, positive or negative.

It is to be remarked in the first place that n can, without loss of generality, be regarded as positive. For, if n were negative, say $n = -n'$, the integrand could be written

$$x^m\left(a + \frac{b}{x^{n'}}\right)^p = x^m\left(\frac{ax^{n'} + b}{x^{n'}}\right)^p = x^{m-pn'}(b + ax^{n'})^p$$

This expression, which is of the same type as $x^m(a + bx^n)^p$, is such that the exponent of x inside the parentheses is positive.

It will now be proved that *an integral of the type* (1) *can in general be reduced to one of the four integrals*

(a) $A\int x^{m-n}(a+bx^n)^p dx$, (b $A\int x^{m+n}(a+bx^n)^p dx$,

(c) $A\int x^m(a+bx^n)^{p-1}dx$, (d) $A\int x^m(a+bx^n)^{p+1}dx$,

plus an algebraic term of the form

$$Bx^\lambda(a+bx^n)^\mu.$$

Here A, B, λ, μ are certain constants which will be determined presently.

Observe that in each of the four cases the integral to which (1) is reduced is of the same type as (1), but that certain changes have taken place in the exponents, viz.,

the exponent m of the monomial factor is increased or diminished by n,

or, the exponent p of the binomial is increased or diminished by unity.

The values of λ and μ are determined by the following rule:

Compare the exponents of the monomial factors in the given integral and in the integral to which it is to be reduced. Select the less of the two numbers and increase it by unity. The result is the value of λ. *In like manner, compare the exponents of the binomial factors in the two integrals, select the less, and increase it by unity. This gives* μ.

Thus, if it is desired to reduce the given integral to

$$A\int x^{m-n}(a+bx^n)^p dx,$$

first write down the formula

$$\int x^m(a+bx^n)^p dx = A\int x^{m-n}(a+bx^n)^p dx + Bx^\lambda(a+bx^n)^\mu.$$

The exponents of the monomial factors in the two integrals are m and $m-n$ respectively, of which $m-n$ is the less. This, increased by unity, gives the value of λ; that is, $\lambda = m - n + 1$.

Again, the exponent of the binomial factor in each integral is the same, namely p, so that there is no choice as to which of the two is the less. Increase this number p by unity to obtain the value of μ. Hence $\mu = p + 1$.

The above formula may now be written

$$\int x^m(a+bx^n)^p dx = A\int x^{m-n}(a+bx^n)^p dx + Bx^{m-n+1}(a+bx^n)^{p+1}. \quad (2)$$

In order to determine the values of the unknown constants A and B, simplify the equation by differentiating both members. After being divided by $x^{m-n}(a+bx^n)^p$ the resulting equation is reduced to

$$x^n = A + Ba(m-n+1) + Bb(m+np+1)x^n.$$

By equating coefficients of like powers of x in both members, we find the values of A and B to be

$$A = -\frac{a(m-n+1)}{b(m+np+1)}, \quad B = \frac{1}{b(m+np+1)}.$$

When these values are substituted in formula (2), it becomes

$$\int x^m(a+bx^n)^p dx = -\frac{a(m-n+1)}{b(m+np+1)}\int x^{m-n}(a+bx^n)^p dx + \frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(m+np+1)}. \quad [\mathbf{A}]$$

Notice that the existence of formula (2) has been proved by showing that values can be found for A and B which make the two members of this equation identical.

There is one case, however, in which this reduction is impossible, viz., when

$$m + np + 1 = 0,$$

for in that case A and B become infinite. [See Ex. 4, p. 235.]

In a similar manner the three following formulas may be derived:

$$\int x^m(a+bx^n)^p\,dx$$

$$= -\frac{b(m+n+np+1)}{a(m+1)}\int x^{m+n}(a+bx^n)^p dx + \frac{x^{m+1}(a+bx^n)^{p+1}}{a(m+1)}. \quad \textbf{[B]}$$

$$\int x^m(a+bx^n)^p\,dx$$

$$= \frac{anp}{m+np+1}\int x^m(a+bx^n)^{p-1}dx + \frac{x^{m+1}(a+bx^n)^p}{m+np+1}. \quad \textbf{[C]}$$

$$\int x^m(a+bx^n)^p\,dx$$

$$= \frac{m+n+np+1}{an(p+1)}\int x^m(a+bx^n)^{p+1}dx - \frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)}. \quad \textbf{[D]}$$

The cases in which the above reductions are impossible are,

For formulas [**A**] and [**C**], when $m + np + 1 = 0$;
for formula [**B**], when $m + 1 = 0$;
for formula [**D**], when $p + 1 = 0$.

Ex. 1. $\int x^3\sqrt{a^2 - x^2}\,dx.$

If the monomial factor were x instead of x^3, the integration could easily be effected by using formula **I.** Since in the present case $m = 3$, $n = 2$, formula [**A**], which diminishes m by n, will reduce the above integral to one that can be directly integrated.

Instead of substituting in [A], as might readily be done, it is best to apply to particular problems the same mode of procedure that was used in deriving the general formula. There are two advantages in this. First, it makes the student independent of the formulas, and second, when several reductions have to be made in the same problem, the work is generally shorter. [See Ex. 4.]

Accordingly assume

$$\int x^3(a^2 - x^2)^{\frac{1}{2}}\, dx = A\int x(a^2 - x^2)^{\frac{1}{2}}\, dx + Bx^2(a^2 - x^2)^{\frac{3}{2}},$$

the values of λ and μ having been determined by the previously given rule.

Differentiate, and divide the resulting equation by $x(a^2 - x^2)^{\frac{1}{2}}$. This gives

$$x^2 = A + B(2\,a^2 - 5\,x^2),$$

from which, on equating coefficients of like powers of x,

$$A = \frac{2\,a^2}{5},\ \ B = -\frac{1}{5},$$

hence,

$$\int x^3\sqrt{a^2 - x^2}\, dx = \frac{2\,a^2}{5}\int (a^2 - x^2)^{\frac{1}{2}} x\, dx - \tfrac{1}{5}\, x^2(a^2 - x^2)^{\frac{3}{2}}$$

$$= -\tfrac{1}{15}(2\,a^2 + 3\,x^2)(a^2 - x^2)^{\frac{3}{2}}.$$

Ex. **2.** $\int \sqrt{x^2 - 2\,x - 3}\, dx.$

By following the suggestions of Art. 118, this integral can be reduced to the form

$$\int \sqrt{z^2 - 4}\, dz,$$

in which $z = x - 1$.

Assume

$$\int (z^2 - 4)^{\frac{1}{2}} dz = A\int (z^2 - 4)^{-\frac{1}{2}}\, dz + Bz(z^2 - 4)^{\frac{1}{2}}.$$

In determining λ notice that $m = 0$ in both integrals, so that $\lambda = 0 + 1 = 1$. Also, $\mu = -\tfrac{1}{2} + 1 = \tfrac{1}{2}$.

Ex. 3. $\int \sqrt{2\,ax - x^2}\,dx.$

The mode of procedure of Ex. 2 may be followed. Another method can also be used, as follows.

On writing in the form

$$\int x^{\frac{1}{2}}(2\,a - x)^{\frac{1}{2}}\,dx,$$

and observing that the integration of

$$\int x^{-\frac{1}{2}}(2\,a - x)^{-\frac{1}{2}}\,dx = \int \frac{dx}{\sqrt{2\,ax - x^2}}$$

can be performed (see Ex. 10, p. 222), it will be seen that integration may be effected in the present case by reducing each of the exponents m and p by unity. This is possible since $n = 1$ and m can accordingly be diminished by 1. Hence assume

$$\int x^{\frac{1}{2}}(2\,a - x)^{\frac{1}{2}}\,dx = A'\int x^{-\frac{1}{2}}(2\,a - x)^{\frac{1}{2}}\,dx + B'x^{\frac{1}{2}}(2\,a - x)^{\frac{3}{2}}.$$

The exponent of the binomial in the new integral may be reduced in turn by assuming

$$\int x^{-\frac{1}{2}}(2\,a - x)^{\frac{1}{2}}\,dx = A''\int x^{-\frac{1}{2}}(2\,a - x)^{-\frac{1}{2}}\,dx + B''\,x^{\frac{1}{2}}(2\,a - x)^{\frac{1}{2}}.$$

When this expression is substituted for the integral in the second member of the preceding equation, the result takes the form

$$\int \sqrt{2\,ax - x^2}\,dx = A\int \frac{dx}{\sqrt{2\,ax - x^2}} + Bx^{\frac{1}{2}}(2\,a - x)^{\frac{1}{2}} + Cx^{\frac{1}{2}}(2\,a - x)^{\frac{3}{2}},$$

in which A, B, C are written for brevity in the place of $A'A''$, $A'B''$, B' respectively. The values of A, B, C are calculated in the usual manner by differentiating, simplifying, and equating coefficients of like powers of x.

The method just given requires two reductions, and hence is less suitable than that employed in Ex. 2, which requires but one reduction.

The rule for determining the values of λ and μ may now be advantageously abbreviated. Let m, p be the exponents of the two factors in the given integral, and m', p' the corresponding exponents in the new integral. Of these two pairs, m, p and m', p', one of the numbers in the one pair is less than the corresponding number in the other pair. This fact will be expressed briefly by saying that the one pair is less than the other pair. With this understanding the preceding rule may be expressed as follows:

Select the less of the two pairs of exponents m, p and m', p'. Increase each number in the pair selected by unity. This gives the pair of exponents λ, μ.

Ex. 4. $\displaystyle\int\frac{x^4\,dx}{(x^2+a^2)^{\frac{3}{2}}}$.

Assume successively

$$\int x^4(x^2+a^2)^{-\frac{3}{2}}\,dx = A'\int x^4(x^2+a^2)^{-\frac{1}{2}}\,dx + B'x^5(x^2+a^2)^{-\frac{1}{2}},$$

$$\int x^4(x^2+a^2)^{-\frac{1}{2}}\,dx = A''\int x^2(x^2+a^2)^{-\frac{1}{2}}\,dx + B''x^3(x^2+a^2)^{\frac{1}{2}},$$

$$\int x^2(x^2+a^2)^{-\frac{1}{2}}\,dx = A'''\int (x^2+a^2)^{-\frac{1}{2}}\,dx + B'''x(x^2+a^2)^{\frac{1}{2}}.$$

These equations may be combined into the single formula

$$\int x^4(x^2+a^2)^{-\frac{3}{2}}\,dx = A\int (x^2+a^2)^{-\frac{1}{2}}\,dx + Bx(x^2+a^2)^{\frac{1}{2}}$$
$$+ Cx^3(x^2+a^2)^{\frac{1}{2}} + Dx^5(x^2+a^2)^{-\frac{1}{2}}.$$

The values of the coefficients are found to be

$$A = -\tfrac{3}{2}a^2,\quad B = \frac{3}{2},\quad C = -\frac{1}{a^2},\quad D = \frac{1}{a^2}.$$

Hence

$$\int x^4(x^2+a^2)^{-\frac{3}{2}}\,dx = \frac{x^3+3a^2x}{2\sqrt{x^2+a^2}} - \tfrac{3}{2}a^2\log\left(x+\sqrt{x^2+a^2}\right).$$

In this example three reductions were necessary; first, a reduction of type [***D***], second, and third, a reduction of type [***A***]. Can these reductions be taken in any order?

The different possible arrangements of the order in which these three reductions might succeed each other are

(1) [***A***], [***A***], [***D***]; (2) [***A***], [***D***], [***A***]; (3) [***D***], [***A***], [***A***],

of which number (3) was chosen in the solution of the problem. Of the other two arrangements, (2) can be used, but (1) cannot. For, after first applying [***A***] (which would be done in either case), the new integral is

$$\int x^2(a^2+x^2)^{-\frac{3}{2}}\,dx.$$

If [***A***] were now applied it would be necessary to assume

$$\int x^2(a^2+x^2)^{-\frac{3}{2}}\,dx = A\int (a^2+x^2)^{-\frac{3}{2}} + Bx(a^2+x^2)^{-\frac{1}{2}}.$$

This equation, when differentiated and simplified, becomes

$$x^2 = A + Ba^2,$$

a relation which it is clearly impossible to reduce to an identity by equating coefficients of like powers of x, since there is no x^2 term in the right member to correspond with the one in the left member. It will be observed that this is the exceptional case mentioned on page 232, in which $m + np + 1 = 0$.

EXERCISES

1. $\int (a^2-x^2)^{\frac{3}{2}}\,dx.$

2. $\int \dfrac{dx}{(x^2+4)^2}.$

3. $\int \dfrac{dx}{(x^2-x+1)^2}.$

4. $\int \dfrac{x^2\,dx}{\sqrt{a^2-x^2}}.$

5. $\int \sqrt{a^2-x^2}\,dx.$

6. $\int \dfrac{dx}{x^2\sqrt{a^2-x^2}}.$

7. $\int \dfrac{dx}{(x^2+a)^{\frac{5}{2}}}.$

8. $\int (a^2+x^2)^{\frac{3}{2}}\,dx.$

9. $\int \sqrt{x^2 + a}\, dx.$

10. $\int x\sqrt{2\,ax - x^2}\, dx.$

11. $\int \frac{\sqrt{2ax - x^2}}{x^3}\, dx.$

12. $\int \frac{dx}{\sqrt{x^6 + x^{10}}}.$

13. $\int \frac{dx}{(x^2 + 4\,x + 3)^3}.$

14. $\int \sqrt{1 - 2\,x - x^2}\, dx.$

15. Show that

$$\int \frac{dx}{(x^2 + c)^n} = \frac{1}{2\,c(n-1)}\left[\frac{x}{(x^2 + c)^{n-1}} + (2\,n - 3)\int \frac{dx}{(x^2 + c)^{n-1}}\right].$$

16. $\int \frac{dx}{x^3\sqrt{x^2 - 1}}.$

17. $\int \frac{x\, dx}{(x^2 + 7)^2}.$

18. $\int \frac{dx}{(a + bx^2)^{\frac{3}{2}}}.$

19. $\int \frac{\sin\theta\, d\theta}{(1 + e\sin^2\theta)^{\frac{3}{2}}}.$

[Substitute $\cos\theta = x$.]

20. $\int x^2\sqrt{a^2 - x^2}\, dx.$

21. $\int (a^2 - x^2)^{\frac{5}{2}}\, dx.$

CHAPTER III

INTEGRATION OF RATIONAL FRACTIONS

121. Decomposition of rational fractions. The object of the present chapter is to show how to integrate fractions of the form

$$\frac{\phi(x)}{\psi(x)},$$

wherein $\phi(x)$ and $\psi(x)$ are polynomials in x.

The desired result is accomplished by the method of separating the given fraction into a sum of terms of a simpler kind, and integrating term by term.

If the degree of the numerator is equal to or greater than the degree of the denominator, the indicated division can be carried out until a remainder is obtained which is of lower degree than the denominator. Hence the fraction can be reduced to the form

$$\frac{\phi(x)}{\psi(x)} = ax^n + bx^{n-1} + \cdots + \frac{f(x)}{\psi(x)},$$

in which the degree of $f(x)$ is less than that of $\psi(x)$.

As to the remainder fraction $\frac{f(x)}{\psi(x)}$, it is to be remarked in the first place that the methods of the preceding articles are sufficient to effect the integration of such simple fractions as

$$\frac{A}{x-a}, \frac{A'}{(x-a)^2}, \cdots; \quad \frac{Mx+n}{x^2 \pm a^2}, \frac{M'x+N'}{(x^2 \pm a^2)^2}, \cdots; \quad \frac{Px+Q}{x^2+mx+n}, \cdots. \quad (1)$$

Now the sum of several such fractions is a fraction of the kind under consideration, viz., one whose numerator is of lower degree than its denominator. The question naturally arises as to whether the converse is possible, that is: *Can every fraction* $\frac{f(x)}{\psi(x)}$ *be separated into a sum of fractions of as simple types as those given in* (1)*?*

The answer is, yes.

Since the sum of several fractions has for its denominator the least common multiple of the several denominators, it follows that if $\frac{f(x)}{\psi(x)}$ can be separated into a sum of simpler fractions, the denominators of these fractions must be divisors of $\psi(x)$. Now it is known from Algebra that *every polynomial* $\psi(x)$ *having real coefficients* (and only those having real coefficients are to be considered in what follows) *is the product of factors of either the first or the second degree, the coefficients of each factor being real.*

This fact naturally leads to the discussion of four different cases.

I. When $\psi(x)$ can be separated into real factors of the first degree, no two alike.

E.g., $$\psi(x) = (x - a)(x - b)(x - c).$$

II. When the real factors are all of the first degree, some of which are repeated.

E.g., $$\psi(x) = (x - a)(x - b)^2(x - c)^3.$$

III. When some of the factors are necessarily of the second degree, but no two such are alike.

E.g., $$\psi(x) = (x^2 + a^2)(x^2 + x + 1)(x - b)(x - c)^2.$$

IV. When second degree factors occur, some of which are repeated.

E.g., $$\psi(x) = (x^2 + a^2)^2(x^2 - x + 1)(x - b).$$

122. Case I. **Factors of the first degree, none repeated.** When $\psi(x)$ is of the form

$$\psi(x)=(x-a)(x-b)(x-c)\cdots(x-n),$$

assume
$$\frac{f(x)}{\psi(x)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\cdots+\frac{N}{x-n},$$

in which $A, B, C, \cdots, N$ are constants whose values are to be determined by the condition that the sum of the terms in the right-hand member shall be identical with the left-hand member.

Ex. $\displaystyle\int\frac{x^3-3x^2+x}{x^2-3x+2}\,dx.$

Dividing numerator by denominator, we obtain

$$\frac{x^3-3x^2+x}{x^2-3x+2}=x-\frac{x}{x^2-3x+2}.$$

Assume
$$\frac{x}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}.$$

By clearing of fractions, we have

(1) $$x=A(x-2)+B(x-1).$$

In order that the two members of this equation may be identical it is necessary that the coefficients of like powers of x be the same in each.

Hence $\qquad 1=A+B,\quad 0=-2A-B,$

from which $\qquad A=-1,\ B=2.$

Accordingly the given integral becomes

$$\int\left(x+\frac{1}{x-1}-\frac{2}{x-2}\right)dx=\frac{x^2}{2}+\log(x-1)-2\log(x-2)$$
$$=\frac{x^2}{2}+\log\frac{x-1}{(x-2)^2}.$$

A shorter method of calculating the coefficients can be used. Since equation (1) is an identity, it is true for all values of x. By giving x the value 1 the equation reduces to $1 = A(-1)$, or $A = -1$. Again, assume $x = 2$. Whence $2 = B$.

EXERCISES

1. $\displaystyle\int \frac{dx}{x^2 - a^2}$.

2. $\displaystyle\int \frac{1 - 3x}{x^3 - x} dx$.

3. $\displaystyle\int \frac{(x^3 - 12)\,dx}{x^2 + 4x + 3}$.

4. $\displaystyle\int \frac{(x^2 - ab)\,dx}{(x - a)(x - b)}$.

5. $\displaystyle\int \frac{x\,dx}{x^2 - 4x + 1}$.

6. $\displaystyle\int \frac{(x^2 - 1)\,dx}{(x^2 - 4)(4x^2 - 1)}$.

7. $\displaystyle\int \frac{x^2 - 2cx + ac - ab + bc}{(x - a)(x - b)(x - c)} dx$.

8. $\displaystyle\int x^2(x + a)^{-1}(x + b)^{-1}\,dx$.

9. $\displaystyle\int \frac{(3x + 1)\,dx}{2x^2 + 3x - 2}$.

10. $\displaystyle\int \frac{(x^2 + ab)\,dx}{x(x - a)(x + b)}$.

11. $\displaystyle\int \frac{(x + 4)\,dx}{2x - x^2 - x^3}$.

12. $\displaystyle\int \frac{x^3\,dx}{x^2 + 7x + 12}$.

13. $\displaystyle\int \frac{dx}{a^2x^2 - b^2}$.

14. $\displaystyle\int \frac{\sec^2 x\,dx}{1 - \tan^2 x}$.

[Put $\tan x = t$.]

123. Case II. Factors of the first degree, some repeated.

Ex. $\displaystyle\int \frac{(5x^2 - 3x + 1)\,dx}{x(x - 1)^3}$.

Assume

$$\frac{5x^2 - 3x + 1}{x(x - 1)^3} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3}. \tag{1}$$

To justify this assumption, observe that:

(a) In adding the fractions in the right-hand member, the least common multiple of the denominators will be $x(x - 1)^3$, which is identical with the denominator in the left-hand member.

(*b*) Further, the expressions x, $x-1$, $(x-1)^2$, $(x-1)^3$ are the only ones which can be assumed as denominators of the partial fractions, since these are the only divisors of $x(x-1)^3$ consisting of powers of a prime factor.

(*c*) When equation (1) is cleared of fractions, and the coefficients of like powers of x in both members are equated, four equations are obtained, exactly the right number from which to determine the four unknown constants A, B, C, D.

Instead of the method just indicated in (*c*) for calculating the coefficients, a more rapid process would be as follows.

By clearing of fractions, the identity (1) may be written

$$5x^2 - 3x + 1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx.$$

Putting $x = 1$ gives at once $3 = D$.

Substitute for D the value just found, and transpose the corresponding term. This gives

$$5x^2 - 6x + 1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1).$$

It can be seen by inspection that the right-hand member of the result is divisible by $x-1$. As this relation is an identity, it follows that the left-hand member is also divisible by $x-1$. When this factor is removed from both members, the equation reduces to

$$5x - 1 = A(x-1)^2 + Bx(x-1) + Cx.$$

Now put $x = 1$. Then $C = 4$.

Substitute the value found for C, transpose, and divide by $x-1$. The result is $$1 = A(x-1) + Bx.$$

By giving x the values 0 and 1 in succession, we find that

$$A = -1, \quad B = 1.$$

Accordingly, we have

$$\int \frac{(5x^2 - 3x + 1)dx}{x(x-1)^3} = \int \left(-\frac{1}{x} + \frac{1}{x-1} + \frac{4}{(x-1)^2} + \frac{3}{(x-1)^3} \right) dx$$

$$= \log \frac{x-1}{x} - \frac{8x-5}{2(x-1)^2}.$$

EXERCISES

1. $\displaystyle\int\frac{dx}{(x-1)^2(x+1)}$.

2. $\displaystyle\int\frac{dx}{x^3(x-1)}$.

3. $\displaystyle\int\frac{x\,dx}{(x^2-a^2)^2}$.

4. $\displaystyle\int\frac{(\sqrt{2}\,x+1)dx}{x^2(x+\sqrt{2})^2}$.

5. $\displaystyle\int\frac{x^5-5x-3}{x^2(x+1)^2}dx$.

6. $\displaystyle\int\frac{2(x^3+a^2x)dx}{x^4-2a^2x^2+a^4}$.

7. $\displaystyle\int\frac{\sqrt{2}\,dx}{(2+\sqrt{2}-\sqrt{2}\,x)^3}$.

8. $\displaystyle\int\frac{ax^3+a^2x^2+(a+1)x+a}{x^2(a+x)}dx$.

9. $\displaystyle\int\frac{(x^3-1)dx}{x^3+3x^2}$.

10. $\displaystyle\int(ax^2+bx^3)^{-1}\,dx$.

11. $\displaystyle\int\frac{(x^2-11x+26)dx}{(x-3)^2}$.

[Substitute $x-3=z$.]

12. $\displaystyle\int\frac{x^2\,dx}{(x-a)^3}$.

[Substitute $x-a=z$.]

124. Case III. **Occurrence of quadratic factors, none repeated.**

Ex. 1. $\displaystyle\int\frac{(4x^2+5x+4)dx}{(x^2+1)(x^2+2x+2)}$.

Assume

$$(1)\qquad \frac{4x^2+5x+4}{(x^2+1)(x^2+2x+2)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2x+2}.$$

Then

$$(2)\quad 4x^2+5x+4=(Ax+B)(x^2+2x+2)+(Cx+D)(x^2+1).$$

By equating coefficients of like powers of x

$$0=A+C,\qquad 5=2A+2B+C,$$

$$4=2A+B+D,\qquad 4=2B+D,$$

from which $\qquad A=1,\ B=2,\ C=-1,\ D=0.$

Hence the given integral becomes

$$\int\frac{(x+2)dx}{x^2+1}-\int\frac{x\,dx}{x^2+2x+2}=2\tan^{-1}x+\tan^{-1}(x+1)+\tfrac{1}{2}\log\frac{x^2+1}{x^2+2x+2}.$$

To make clear the reasons for the assumption which was made concerning the form of equation (1), observe that since the factors of the denominator in the left member are $x^2 + 1$ and $x^2 + 2x + 2$, these must necessarily be the denominators in the right member. Also, since the numerator of the given fraction is of lower degree than its denominator, the numerator of each partial fraction must be of lower degree than its denominator. As the latter is of the second degree in each case, the most general form for a numerator fulfilling this requirement (*i.e.*, to be of lower degree than its denominator) is an expression of the first degree such as $Ax + B$, or $Cx + D$.

Notice, besides, that in equating the coefficients of like powers of x in opposite members of equation (2), four equations are obtained which exactly suffice to determine the four unknown coefficients A, B, C, D.

Ex. **2.** $\displaystyle\int \frac{dx}{(x^2+1)(x^2+2)}$.

We can assume in this case $\displaystyle\frac{1}{(x^2+1)(x^2+2)} = \frac{A}{x^2+1} + \frac{B}{x^2+2}$.

For if we make the substitution $x^2 = t$, the given fraction becomes $\displaystyle\frac{1}{(t+1)(t+2)}$, to which Case I is applicable.

EXERCISES

1. $\displaystyle\int \frac{4\,dx}{x^3+4x}$.

2. $\displaystyle\int \frac{x\,dx}{(x+1)(x^2+1)}$.

3. $\displaystyle\int \frac{dx}{x^3+a^3}$.

4. $\displaystyle\int \frac{(a^2-b^2)\,dx}{(x^2+a^2)(x^2+b^2)}$.

5. $\displaystyle\int \frac{(4x-6)\,dx}{x^4+2x^2}$.

6. $\displaystyle\int \frac{x\,dx}{x^4+x^2+1}$.

7. $\displaystyle\int \frac{x\,dx}{(x-a)^2(x^2+a^2)}$.

8. $\displaystyle\int \frac{(x^3+2x+2)\,dx}{(x-1)(x^2+2x+2)}$.

9. $\displaystyle\int \frac{2\,dx}{(x-1)(x^2+1)}$.

125. CASE IV. **Occurrence of quadratic factors, some repeated.** This case bears the same relation to Case III that Case II bears to Case I, and an exactly analogous mode of procedure is to be followed.

Ex. $\displaystyle\int\frac{2x^5 - x^4 + 8x^3 + 4}{(x^2+2)^3}dx.$

Assume

$$\frac{2x^5 - x^4 + 8x^3 + 4}{(x^2+2)^3} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)^3}.$$

Whence, by clearing of fractions,

$$2x^5 - x^4 + 8x^3 + 4 = (Ax+B)(x^2+2)^2 + (Cx+D)(x^2+2) + Ex + F. \quad (1)$$

Instead of equating coefficients of like powers of x, as might be done, we may calculate the values of A, B, C, $\cdots$ by the following briefer method.

Substitute for x^2 the value -2, or, what is the same thing, let $x = \sqrt{-2}$. This causes all the terms of the right member to drop out except the last two, and equation (1) reduces to

$$-8\sqrt{-2} = E\sqrt{-2} + F.$$

By equating real and imaginary terms in both members, we obtain

$$-8 = E, \ 0 = F.$$

Substitute the values found for E and F in (1), and transpose the corresponding terms. Both members will then contain the factor x^2+2. On striking this out the equation reduces to

$$2x^3 - x^2 + 4x + 2 = (Ax+B)(x^2+2) + Cx + D.$$

Proceed as before by putting $x^2 = -2$. Whence

$$4 = C\sqrt{-2} + D,$$

and therefore $\qquad 0 = C, \ 4 = D.$

Substitute these values, transpose, and divide by $x^2 + 2$. This gives

$$2x - 1 = Ax + B,$$

whence $$A = 2,\ B = -1.$$

The given integral accordingly reduces to

$$\int \frac{2x-1}{x^2+2}\,dx + \int \frac{4\,dx}{(x^2+2)^2} - \int \frac{8x\,dx}{(x^2+2)^3}.$$

The first term becomes

$$\int \frac{2x\,dx}{x^2+2} - \int \frac{dx}{x^2+2} = \log(x^2+2) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}.$$

The second, integrated by the method of reduction (Chap. II), gives

$$\frac{x}{x^2+2} + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}.$$

Finally, by using formula **I** the last term is integrated immediately. Hence

$$\int \frac{2x^5 - x^4 + 8x^3 + 4}{(x^2+2)^3}\,dx = \log(x^2+2) + \frac{x}{x^2+2} + \frac{2}{(x^2+2)^2}.$$

EXERCISES

1. $\int \left(\frac{x-1}{x^2+1}\right)^2 dx.$

2. $\int \frac{(x+a)^2 + a^2}{(x^2+a^2)^2}\,dx.$

3. $\int \frac{2x\,dx}{(1+x)(1+x^2)^2}.$

4. $\int \frac{(-3x+2)\,dx}{x^2(x^2+1)^2}.$

5. $\int \frac{2x^3 + 2a^2x - x^2}{(x^2+a^2)^2}\,dx.$

6. $\int \frac{x^5\,dx}{(1+x^2)^3}.$

[Ex. 6 can also be integrated, and more easily, by means of the substitution $1 + x^2 = t$.]

The principles used in the preceding cases in the assumption of the partial fractions may be summed up as follows:

Each of the denominators of the partial fractions contains one and only one of the prime factors of the given denominator.

When a prime factor occurs to the nth power in the denominator of the given fraction, all of its different powers from the first to the nth must be used as denominators of the partial fractions.

The numerator of each of the assumed fractions is of degree one lower than the degree of the prime factor whose power occurs in the corresponding denominator.

126. General theorem. Since every rational fraction can be integrated by first separating it, if necessary, into simpler fractions in accordance with some one of the cases considered above, the important conclusion is at once deducible:

The integral of every rational algebraic fraction is expressible in terms of algebraic, logarithmic, and inverse-trigonometric functions.

CHAPTER IV

INTEGRATION BY RATIONALIZATION

At the end of the preceding chapter it was remarked that every rational algebraic function can be integrated. The question as to the possibility of integrating irrational functions has next to be considered. This has already been touched upon in Chapter II, where a certain type of irrational functions was treated by the method of reduction.

In the present chapter it is proposed to consider the simplest cases of irrational functions, viz., those containing $\sqrt[n]{ax+b}$ and $\sqrt{ax^2+bx+c}$, and to show how, by a process of rationalization, every such function can be integrated.

127. Integration of functions containing the irrationality $\sqrt{ax+b}$. When the integrand contains $\sqrt[n]{ax+b}$, that is, the nth root of an expression of the first degree in x, but no other irrationality, it can be reduced to a rational form by means of the substitution

$$\sqrt[n]{ax+b}=z.$$

Ex. 1. $\displaystyle\int\frac{dx}{\sqrt{2x+3}-1}$.

Assume $\qquad \sqrt{2x+3}=z,$

that is, $\qquad 2x+3=z^2.$

Then $\qquad dx=z\,dz,$

and
$$\int\frac{dx}{\sqrt{2x+3}-1}=\int\frac{z\,dz}{z-1}=z+\log(z-1)$$
$$=\sqrt{2x+3}+\log(\sqrt{2x+3}-1).$$

Ex. 2. $\displaystyle\int \frac{1 + x^{\frac{1}{6}} - x^{\frac{1}{3}} - \sqrt{x}}{x^{\frac{2}{3}} + x}\,dx.$

It would appear at first sight that this integrand contains several irrationalities, viz., $\sqrt{x}$, $\sqrt[3]{x}$, $\sqrt[6]{x}$. It is readily seen, however, that they are all powers of $\sqrt[6]{x}$, and hence the substitution $\sqrt[6]{x} = z$ will rationalize the expression to be integrated.

EXERCISES

1. $\displaystyle\int \frac{dx}{x\sqrt{x+1}}.$

2. $\displaystyle\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}.$

3. $\displaystyle\int \frac{dx}{2\sqrt{x-1} + x}.$

4. $\displaystyle\int \frac{dx}{(x-1)\sqrt{x-2}}.$

5. $\displaystyle\int \frac{dx}{(x-a-b^2)\sqrt{x-a}}.$

6. $\displaystyle\int \frac{x^{\frac{1}{7}} + \sqrt{x}}{x^{\frac{8}{7}} + x^{\frac{15}{14}}}\,dx.$

When two irrationalities of the form $\sqrt{ax+b}$, $\sqrt{cx+d}$ occur in the integrand, the first radical can be made to disappear by the substitution

$$\sqrt{ax+b} = z.$$

The second radical then reduces to

$$\sqrt{\frac{c}{a}(z^2 - b) + d},$$

and the method of the next article can be applied.

128. Integration of expressions containing $\sqrt{ax^2 + bx + c}$. Every expression containing $\sqrt{ax^2 + bx + c}$, but no other irrationality, can be rationalized by a proper substitution. Two cases are distinguished.

(*a*) *When $ax^2 + bx + c$ has real factors.* We may then write the quadratic expression in the factored form

$$ax^2 + bx + c = a(x - \alpha)(x - \beta), \tag{1}$$

in which α and β are real. Introduce a new variable t by means of the formula

$$\sqrt{ax^2+bx+c}=t(x-\alpha). \tag{A}$$

Square both members of this equation and replace the left member by means of (1). This gives

$$a(x-\alpha)(x-\beta)=t^2(x-\alpha)^2.$$

On canceling $x-\alpha$ and solving for x we obtain as the equation of transformation

$$x=\frac{\alpha t^2-a\beta}{t^2-a}. \tag{2}$$

Hence x (and therefore dx) is rationally expressible in terms of t, while the radical reduces to

$$t\left[\frac{\alpha t^2-a\beta}{t^2-a}-\alpha\right]=\frac{at(\alpha-\beta)}{t^2-a}, \tag{3}$$

which is also rational in t. The substitution of these expressions in the proposed integrand gives a rational fraction which may be treated by the methods of the preceding chapter.

(*b*) *When a, the coefficient of x^2, is positive.*

Make the substitution

$$\sqrt{ax^2+bx+c}=\sqrt{a}\cdot x+t. \tag{B}$$

By squaring both members and solving for x we obtain

$$x=\frac{t^2-c}{b-2\sqrt{a}t}, \tag{4}$$

while the radical is expressible in the form

$$\frac{\sqrt{a}t^2-bt+\sqrt{a}c}{2\sqrt{a}t-b}, \tag{5}$$

and hence the integrand becomes rational when expressed in terms of t.

The only case that is not included in (a) or (b) is that in which the factors of $ax^2 + bx + c$ are imaginary and the coefficient a is negative; the radical is then imaginary for all values of x. Although the integral can be obtained (in an imaginary form) by either of the preceding substitutions, this case does not arise in practical applications of the calculus and will not be considered further.

Ex. 1. $$\int \frac{dx}{x + \sqrt{x^2 + 2x - 1}}.$$

Formula (**B**) gives

$$\sqrt{x^2 + 2x - 1} = x + t,$$

whence, by solving for x, we obtain

$$x = \frac{t^2 + 1}{2(1 - t)},$$

and accordingly $$dx = \frac{-t^2 + 2t + 1}{2(1 - t)^2}\,dt,$$

$$\sqrt{x^2 + 2x - 1} = \frac{-t^2 + 2t + 1}{2(1 - t)}.$$

When these expressions are substituted in the above integral it reduces to

$$\int \frac{(-t^2 + 2t + 1)\,dt}{2(1 + t)^2}.$$

The work of integrating may be facilitated by means of the transformation $1 + t = z$. The result, in terms of x, is

$$\tfrac{1}{2}(x - \sqrt{x^2 + 2x - 1}) + \frac{1}{1 - x + \sqrt{x^2 + 2x - 1}}$$
$$+ 2 \log(1 - x + \sqrt{x^2 + 2x - 1}).$$

Ex. **2.** $\displaystyle\int\frac{\sqrt{1+x}\,dx}{(1-x)\sqrt{1-x}}.$

By rationalizing either numerator or denominator we obtain $\sqrt{1-x^2}$ as the radical part of the integrand.

Formula (**A**) gives $\quad \sqrt{1-x^2}=t(1-x),$

whence
$$\sqrt{\frac{1+x}{1-x}}=t, \tag{1}$$

or
$$\frac{1+x}{1-x}=t^2, \tag{2}$$

and hence, by differentiation,
$$\frac{2\,dx}{(1-x)^2}=2\,t\,dt. \tag{3}$$

Add 1 to both members of (2) and combine the two terms of the left member. The result is
$$\frac{2}{1-x}=t^2+1. \tag{4}$$

Dividing (3) by (4), we have
$$\frac{dx}{1-x}=\frac{2\,t\,dt}{1+t^2}. \tag{5}$$

Now multiply (1) and (5) together and integrate. We obtain
$$\int\sqrt{\frac{1+x}{1-x}}\cdot\frac{dx}{1-x}=\int\frac{2\,t^2\,dt}{t^2+1}$$
$$=2\sqrt{\frac{1+x}{1-x}}-2\tan^{-1}\sqrt{\frac{1+x}{1-x}}.$$

EXERCISES

1. $\displaystyle\int\frac{dx}{(1-x)(1-\sqrt{1-x^2})}.$

2. $\displaystyle\int\frac{dx}{\sqrt{2\,x^2-3\,x+1}\,[\sqrt{2\,x^2-3\,x+1}+\sqrt{2}(x-1)]}.$

We can rationalize also by means of a trigonometric substitution. First reduce ax^2+bx+c to the form $\pm t^2 \pm k^2$, as in Art. 118, and then make one of the following transformations:

$$\text{In } k^2-t^2 \text{ put } t=k\sin\theta,$$
$$\text{in } t^2-k^2 \text{ put } t=k\sec\theta,$$
$$\text{in } t^2+k^2 \text{ put } t=k\tan\theta.$$

Since $\sqrt{-t^2-k^2}$ is imaginary, we shall exclude this case from consideration.

The resulting trigonometric functions can then be integrated by methods to be explained in the next chapter.

129. There is one case in which a different transformation leads more rapidly to the desired result. If, after reducing the terms under the radical sign to one of the simple forms mentioned in the preceding paragraph, the integrand can be expressed as the product of $t\,dt$ and a function containing only even powers of t, then we may substitute

$$\sqrt{\pm t^2 \pm k^2}=z.$$

For this gives
$$t^2=\pm(z^2\pm k^2)$$
and
$$t\,dt=\pm z\,dz,$$

and hence the integral takes a rational form in z.

EXERCISES ON CHAPTER IV

1. $\displaystyle\int\frac{(-x^3+4x)dx}{(x^2+2)\sqrt{x^2-1}}$.

[Notice that Art. 129 is applicable.]

2. $\displaystyle\int\frac{[(x-a)^{\frac{2}{3}}-1]dx}{2(x-a)^{\frac{4}{3}}-(x-a)^{\frac{2}{3}}}$.

3. $\displaystyle\int\frac{\sqrt{x+1}\,dx}{\sqrt{x+1}+2}$.

4. $\int \frac{dx}{x+\sqrt{x-1}}$.

8. $\int \frac{dx}{x+\sqrt{x^2-1}}$.

5. $\int \frac{x\,dx}{(a+x)^{\frac{1}{3}}}$.

9. $\int \frac{1+\sqrt{x}}{1+\sqrt[3]{x}}\,dx$.

6. $\int \frac{(2-3x^{-\frac{1}{6}})dx}{x-3x^{\frac{5}{6}}+5x^{\frac{2}{3}}}$.

10. $\int \frac{x^3\,dx}{\sqrt{a^2-x^2}}$.

7. $\int \frac{dx}{(x^2+a^2)\sqrt{x^2-a^2}}$.

11. $\int \frac{\sqrt{a^2-x^2}\,dx}{x}$.

[Use trigonometric substitutions in the following exercises.]

12. $\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx$.

15. $\int \frac{dx}{(x^2+a^2)^{\frac{5}{2}}}$.

13. $\int \frac{dx}{x^2\sqrt{x^2-a^2}}$

16. $\int \frac{\sqrt{x^2+a^2}}{x^2}\,dx$.

14. $\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}}$.

17. $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}}$.

CHAPTER V

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

130. In regard to the integration of trigonometric functions, it is to be remarked in the first place that every rational trigonometric function can be rationally expressed in terms of sine and cosine.

It is accordingly evident that such functions can be integrated by means of the substitution

$$\sin x = z.$$

After the substitution has been effected, the integrand may involve the irrationality

$$\sqrt{1 - z^2} [= \cos x].$$

This can be removed by rationalization, as explained in the preceding chapter, or the method of reduction may be employed.

The substitution $\cos x = z$ will serve equally well.

It is usually easier, however, to integrate the trigonometric forms without any such previous transformation to algebraic functions. The following articles treat of the cases of most frequent occurrence.

131. $\displaystyle\int \sec^{2n} x\, dx, \quad \int \csc^{2n} x\, dx.$

In this case n is supposed to be a positive integer.

If $\sec^{2n} x\, dx$ is written in the form

$$\sec^{2n-2} x \cdot \sec^2 x\, dx = (1 + \tan^2 x)^{n-1} d(\tan x),$$

the first integral becomes

$$\int (\tan^2 x + 1)^{n-1} d(\tan x).$$

If $(\tan^2 x + 1)^{n-1}$ is expanded by the binomial formula and integrated term by term, the required result is readily obtained.

In like manner,

$$\int \csc^{2n} x\, dx = \int \csc^{2n-2} x \cdot \csc^2 x\, dx$$

$$= - \int (\cot^2 x + 1)^{n-1} d(\cot x).$$

This last form can be integrated, as in the preceding case, by expanding the binomial in the integrand.

The same method will evidently apply to integrals of the form

$$\int \tan^m x \sec^{2n} x\, dx, \quad \int \cot^m x \csc^{2n} x\, dx,$$

in which m is any number.

EXERCISES

1. $\displaystyle\int \frac{dx}{\cos^4 x}$.

2. $\displaystyle\int \csc^4 x\, dx$.

3. $\displaystyle\int \sec^6 x\, dx$.

4. $\displaystyle\int \frac{dx}{\sin^8 x \cos^8 x}$.

5. $\displaystyle\int \frac{(1 - \cos x)^2 dx}{\sin^4 x}$.

6. $\displaystyle\int \frac{dx}{\sin^4 x \cos^4 x\, (\cos^4 x - \sin^4 x)^4}$.

7. $\displaystyle\int \frac{dx}{\sin^3 x \cos x} \left[= \int \tan^{-3} x \sec^4 x\, dx\right]$.

8. $\displaystyle\int \frac{\cos^2 x\, dx}{\sin^6 x}$.

132. $\displaystyle\int \sec^m x \tan^{2n+1} x\,dx, \quad \int \csc^m x \cot^{2n+1} x\,dx.$

In these integrands n is a positive integer, or zero, so that $2n+1$ is any positive odd integer, while m is unrestricted.

The first integral may be written in the form

$$\int \sec^{m-1} x \tan^{2n} x \cdot \sec x \tan x\,dx$$

$$= \int \sec^{m-1} x (\sec^2 x - 1)^n d(\sec x),$$

which can be integrated after expanding $(\sec^2 x - 1)^n$ by the binomial formula.

Similarly,

$$\int \csc^m x \cot^{2n+1} x\,dx = \int \csc^{m-1} x \cot^{2n} x \cdot \csc x \cot x\,dx$$

$$= -\int \csc^{m-1} x (\csc^2 x - 1)^n d(\csc x).$$

EXERCISES

1. $\displaystyle\int \sec^2 x \tan^3 x\,dx.$

2. $\displaystyle\int \csc^3 x \cot^5 x\,dx.$

3. $\displaystyle\int \frac{\sec ax}{\cot^5 ax}\,dx.$

4. $\displaystyle\int \sin x \cot^3 x\,dx.$

5. $\displaystyle\int \tan^5 x\,dx.$

6. $\displaystyle\int \frac{\sin^3 x\,dx}{\cos^n x} \left[= \int \sec^{n-3} x \tan^3 x\,dx\right].$

7. $\displaystyle\int \tan x\,dx.$

8. $\displaystyle\int \cot x\,dx.$

133. $\int \tan^n x\, dx, \quad \int \cot^n x\, dx.$

The first integral can be treated thus:

$$\int \tan^n x\, dx = \int \tan^{n-2} \cdot \tan^2 x\, dx$$

$$= \int \tan^{n-2} x(\sec^2 x - 1) dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.$$

When n is a positive integer, the work of integration may be rapidly carried out by writing t for brevity in place of $\tan x$ and then putting $t^n dx$ in a different form by means of the following process. First, divide t^n by t^2+1; the quotient is a polynomial of the form $t^{n-2} - t^{n-4} + t^{n-6} - \cdots$, while the remainder R is either ± 1 or $\pm t$ according as n is even or odd. Then, since the dividend equals the product of divisor and quotient plus the remainder, we have

$$t^n = (t^{n-2} - t^{n-4} + t^{n-6} - \cdots)(t^2+1) + R.$$

But since $\quad (\tan^2 x + 1) dx = \sec^2 x\, dx = d(\tan x) = dt,$

we have

$$\int \tan^n x\, dx = \int (t^{n-2} - t^{n-4} + t^{n-6} - \cdots) dt + \int R\, dx.$$

For example,

$$\int \tan^8 x\, dx = \int (t^6 - t^4 + t^2 - 1) dt + \int dx,$$

and $\quad \int \tan^7 x\, dx = \int (t^5 - t^3 + t)\, dt - \int \tan x\, dx.$

The integral $\int \cot^n x\, dx$ can be treated in a similar manner, in case n is a positive integer.

For any value of n we have

$$\int \cot^n x\, dx = \int \cot^{n-2} x \cot^2 x\, dx$$

$$= \int \cot^{n-2} x (\csc^2 x - 1)\, dx$$

$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx.$$

Since $\tan x$ and $\cot x$ are reciprocals of each other, the above method is sufficient to integrate any integral power of $\tan x$ or $\cot x$.

Another method of procedure would be to make the substitution $\tan x = z$, whence

$$\int \tan^n x\, dx = \int \frac{z^n\, dz}{1 + z^2}.$$

If the exponent n is a fraction, say $n = \frac{p}{q}$, the last integral can be rationalized by the substitution $z = u^q$.

It is evident from this that any rational power of tangent or cotangent can be integrated.

EXERCISES

1. $\int \cot^4 x\, dx.$
2. $\int \tan^3 ax\, dx.$
3. $\int (\tan x - \cot x)^3\, dx.$
4. $\int (\tan^n x + \tan^{n-2} x)\, dx.$
5. $\int \tan^8 x\, dx.$

When n is a positive integer show that

6. $\int \tan^{2n} x\, dx = \frac{\tan^{2n-1} x}{2n-1} - \frac{\tan^{2n-3} x}{2n-3} + \cdots + (-1)^{n-1}(\tan x - x).$

7. $\int \tan^{2n+1} x\, dx = \frac{\tan^{2n} x}{2n} - \frac{\tan^{2n-2} x}{2n-2}$

$$+ \cdots + (-1)^{n-1}\left(\tfrac{1}{2}\tan^2 x + \log \cos x\right).$$

134. $\int \sin^m x \cos^n x\, dx.$

(*a*) **Either m or n a positive odd integer.**

If one of the exponents, for example m, is a positive odd integer, the given integral may be written

$$\int \sin^{m-1} x \cos^n x \sin x\, dx = -\int (1 - \cos^2 x)^{\frac{m-1}{2}} \cos^n x\, d(\cos x).$$

Since m is odd, $m - 1$ is even, and therefore $\frac{m-1}{2}$ is a positive integer. Hence the binomial can be expanded into a finite number of terms, and thus the integration can be easily completed.

Ex. $\int \sin^5 x \sqrt{\cos x}\, dx.$

According to the method just indicated this integral can be reduced to

$$-\int \sin^4 x \sqrt{\cos x}\, d(\cos x) = -\int (1 - \cos^2 x)^2 (\cos x)^{\frac{1}{2}}\, d(\cos x)$$
$$= -\tfrac{2}{3}\cos^{\frac{3}{2}} x + \tfrac{4}{7}\cos^{\frac{7}{2}} x - \tfrac{2}{11}\cos^{\frac{11}{2}} x.$$

EXERCISES

1. $\int \sin^3 x\, dx.$

2. $\int \sin^3 x \cos^4 x\, dx.$

3. $\int \frac{\cos^5 x}{\sin x}\, dx.$

4. $\int \frac{\sin^5 x\, dx}{\cos^2 x \sqrt[3]{\cos x}}.$

5. $\int \frac{\sin^3 x\, dx}{\sqrt{1 - \cos x}}.$

(b) $\boldsymbol{m+n}$ **an even negative integer.**

In this case the integral may be put in the form

$$\int \frac{\sin^m x}{\cos^m x} \cos^{m+n} x \, dx = \int \tan^m x \sec^{-(m+n)} x \, dx,$$

which can be integrated by Art. 131, since the exponent $-(m+n)$ of $\sec x$ is an even positive integer.

Ex. $\displaystyle\int \frac{\sqrt{\sin x}}{\cos^{\frac{9}{2}} x} dx.$

The integration is effected in the following steps:

$$\int \frac{\sqrt{\sin x}\, dx}{\sqrt{\cos x} \cos^4 x} = \int \tan^{\frac{1}{2}} x \sec^4 x \, dx$$

$$= \int \tan^{\frac{1}{2}} x (\tan^2 x + 1) \, d(\tan x)$$

$$= 2 \tan^{\frac{3}{2}} x (^1 + \tfrac{1}{7} \tan^2 x).$$

EXERCISES

1. $\displaystyle\int \frac{\cos^2 x}{\sin^4 x} dx$

2. $\displaystyle\int \frac{dx}{\sin^6 x}.$

3. $\displaystyle\int \frac{\cos^4 x}{\sin^8 x} dx.$

4. $\displaystyle\int \frac{dx}{\sin^4 x \cos^2 x}.$

5. $\displaystyle\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}}.$

6. $\displaystyle\int \frac{\sin^{-2} x}{\cos^{n+2} x} dx.$

(c) **Multiple angles.**

When m and n are both even positive integers, integration may be effected by the use of multiple angles. The trigonometric formulas used for this purpose are

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

$$\cos^2 x = \frac{1 + \cos 2x}{2},$$

$$\sin x \cos x = \frac{\sin 2x}{2}.$$

Ex. $\int \sin^2 x \cos^4 x\, dx.$

$$\begin{aligned}\int \sin^2 x \cos^4 x\, dx &= \int (\sin x \cos x)^2 \cos^2 x\, dx\\ &= \int \frac{\sin^2 2x}{4} \frac{1+\cos 2x}{2} dx\\ &= \frac{1}{8}\int \sin^2 2x\, dx + \frac{1}{16}\int \sin^2 2x \cos 2x\, d(2x)\\ &= \frac{1}{8}\int \frac{1-\cos 4x}{2} dx + \frac{1}{16}\frac{\sin^3 2x}{3}\\ &= \tfrac{1}{16} x - \tfrac{1}{64} \sin 4x + \tfrac{1}{48} \sin^3 2x.\end{aligned}$$

EXERCISES

1. $\int \cos^2 x \sin^2 x\, dx.$

2. $\int \sin^2 x \cos^6 x\, dx.$

3. $\int \sin^4 x \cos^4 x\, dx.$

4. $\int (\sin^4 x - \cos^4 x)^4\, dx.$

5. $\int \frac{\sin^4 x}{\cos^2 x} dx = \int \frac{(1-\cos^2 x)^2}{\cos^2 x} dx = \int (\sec^2 x - 2 + \cos^2 x) dx.$

(*d*) **Reduction formulas.** Integrate $\int \sin^m x \cos^n x\, dx$ by parts,

taking $\quad u = \cos^{n-1}x, \quad dv = \sin^m x \cos x\, dx,$

whence $\quad du = -(n-1)\cos^{n-2}x \sin x\, dx, \quad v = \dfrac{\sin^{m+1}x}{m+1},$

and therefore

$$\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1}x \cos^{n-1}x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2} \cos^{n-2}x\, dx.$$

In the last term replace $\sin^2 x$ by $1-\cos^2 x$ and separate the integral into the two terms

$$\int \sin^m x \cos^{n-2}x\, dx - \int \sin^m x \cos^n x\, dx.$$

Transpose the second integral and unite with the similar integral in the left member. After dividing the resulting equation by $\frac{m+n}{m+1}$ we obtain the formula of reduction

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx$$

by means of which the exponent of the cosine factor may be diminished or increased by 2 according as the integral in the left member or that in the right member is taken as the given integral.

In like manner a reduction formula may be deduced which decreases or increases the exponent of the sine factor by 2. The details are left to the student as an exercise. The result is

$$\int \sin^m x \cos^n x \, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx.$$

The two preceding formulas, when solved for the integrals in the right members, and m (or n) increased by 2, become

$$\int \sin^m x \cos^n x dx = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} \int \sin^m x \cos^{n+2} x dx,$$

$$\int \sin^m x \cos^n x dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} \int \sin^{m+2} x \cos^n x dx.$$

Whenever the values of m and n are such that one of the three preceding cases, (a), (b), (c), is applicable, the integration can generally be performed more quickly by one of those methods.

EXERCISES

1. $\int \sin^2 x\, dx.$ **2.** $\int \frac{\cos^4 x}{\sin^2 x} dx.$

[In Ex. 2 after one reduction, diminishing the exponent of $\cos x$ by 2, Art. 133 may be applied.]

3. $\int \frac{\sin^4 x}{\cos x} dx.$ **4.** $\int \frac{dx}{\sin^3 2x}.$ **5.** $\int \frac{\cos^6 x\, dx}{\sin^4 x}.$

135. $\boldsymbol{\int \frac{dx}{a + b\cos nx}, \int \frac{dx}{a + b\sin nx}, \int \frac{dx}{a + b\sin nx + c\cos nx}.}$

These forms can be integrated by expressing them in terms of the half angle and then in terms of $\tan \frac{nx}{2}$.

Ex. **1.** $\int \frac{dx}{5 + 4\cos x}.$

By making use of the trigonometric relations

$$\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} = 1,$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2},$$

the denominator may be written in the form

$$5\left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}\right) + 4\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right),$$

which becomes $\sin^2 \frac{x}{2} + 9\cos^2 \frac{x}{2}$ on collecting the terms; whence

$$\int \frac{dx}{5 + 4\cos x} = 2\int \frac{\frac{dx}{2}}{\sin^2 \frac{x}{2} + 9\cos^2 \frac{x}{2}}.$$

Now divide numerator and denominator by $\cos^2 \frac{x}{2}$ and bear in mind that $\dfrac{1}{\cos^2\frac{x}{2}} = \sec^2 \frac{x}{2}$. This gives

$$2\int \frac{\sec^2 \frac{x}{2}\, d\left(\frac{x}{2}\right)}{\tan^2 \frac{x}{2} + 9} = \frac{2}{3} \tan^{-1}\left(\frac{1}{3} \tan \frac{x}{2}\right).$$

Ex. 2. $\displaystyle\int \frac{dx}{2 \sin 3x + 1}$.

Express the denominator in the form

$$4 \sin \frac{3x}{2} \cos \frac{3x}{2} + \left(\sin^2 \frac{3x}{2} + \cos^2 \frac{3x}{2}\right).$$

Then, after dividing both terms of the fraction by $\cos^2 \frac{3x}{2}$, the given integral becomes

$$\int \frac{\sec^2 \frac{3x}{2} dx}{\tan^2 \frac{3x}{2} + 4 \tan \frac{3x}{2} + 1}.$$

Now make the substitution $\tan\frac{3x}{2} = t$ and apply Art. 118.

It will be observed from these two problems that the aim is to put the denominator in the form of a *homogeneous quadratic expression* in sine and cosine functions. Then, when both terms of the fraction are divided by the square of the cosine, the denominator becomes quadratic in the tangent function while the numerator can be expressed as the differential of the tangent.

EXERCISES

1. $\displaystyle\int \frac{dx}{5 + 3 \cos 2x}$.

2. $\displaystyle\int \frac{dx}{5 - 3 \sin x}$.

3. $\displaystyle\int \frac{dx}{1 - 2 \sin 2x}$.

4. $\displaystyle\int \frac{dx}{a \sin x + b \cos x}$.

5. $\displaystyle\int \frac{dx}{(a \sin x + b \cos x)^2}$.

6. $\displaystyle\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$.

7. $\displaystyle\int \frac{dx}{1 + \cos^2 x}$.

8. $\displaystyle\int \frac{dx}{1 + \sin x + 2 \cos x}$.

136. $\int e^{ax} \sin nx\, dx$, $\int e^{ax} \cos nx\, dx$.

Integrate $\int e^{ax} \sin nx\, dx$ by parts, assuming

$$u = \sin nx, \text{ and } dv = e^{ax}\, dx.$$

This gives

$$\int e^{ax} \sin nx\, dx = \frac{1}{a} e^{ax} \sin nx - \frac{n}{a}\int e^{ax} \cos nx\, dx. \tag{1}$$

Integrate the same expression again, assuming this time

$$u = e^{ax}, \quad dv = \sin nx\, dx.$$

Then

$$\int e^{ax} \sin nx\, dx = -\frac{1}{n} e^{ax} \cos nx + \frac{a}{n}\int e^{ax} \cos nx\, dx. \tag{2}$$

Multiply (1) by $\frac{a}{n}$ and (2) by $\frac{n}{a}$ and add. The integrals in the right members are eliminated, and the result is

$$\int e^{ax} \sin nx\, dx = \frac{e^{ax}(a \sin nx - n \cos nx)}{a^2 + n^2}.$$

By subtracting (1) from (2), the formula

$$\int e^{ax} \cos nx\, dx = \frac{e^{ax}(n \sin nx + a \cos nx)}{a^2 + n^2}$$

is obtained.

EXERCISES ON CHAPTER V

1. Derive the reduction formula

$$\int \sec^n x\, dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx.$$

[Integrate by parts, taking $u = \sec^{n-2} x$, $dv = \sec^2 x\, dx$.]

2. Derive

$$\int \csc^n x\, dx = -\frac{\cot x \csc^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\, dx.$$

3. $\int \frac{\sqrt{\tan x}\, dx}{\sin x \cos x}$.

4. $\int \frac{\sin x}{\cos^5 x}\, dx$.

5. $\int \frac{dx}{\cos x \sin^2 x}$.

6. $\int \frac{\sin^4 x}{\cos^3 x} dx$.

7. $\int \frac{dx}{(1-x)\sqrt{1-x^2}}$.

[Put $x = \cos\theta$].

8. $\int e^{\frac{x}{2}} \cos\frac{x}{2} dx$.

9. $\int \frac{\sin 2x}{e^x} dx$.

10. $\int e^{2x} \sin^2 x\, dx$.

11. $\int e^x \sin 2x \sin x\, dx$.

[Hint. $2\sin 2x \sin x = \cos x - \cos 3x$.]

12. Show that

$$\int \sin ax \sin bx\, dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}.$$

Use the trigonometric formula

$$\sin\alpha \sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)].$$

13. Show that

$$\int \sin ax \cos bx\, dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}.$$

14. Show that

$$\int \cos ax \cos bx\, dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}.$$

15. $\int \sin^n x \cos^3 x\, dx$.

16. $\int \frac{dx}{\sin^{\frac{3}{2}} x \cos^{\frac{5}{2}} x}$.

17. $\int (\tan x + \cot x)^6 dx$.

18. $\int \frac{dx}{(1+\cos x)^3}$.

19. $\int \frac{dx}{\sin x \cos^3 x - \sin^3 x \cos x}$.

20. $\int \frac{dx}{\sin^4 x \cos^4 x}$.

21. $\int \frac{\sqrt{a^2 - x^2}}{x^4} dx$.

22. $\int \frac{a\sin x + b\cos x}{\alpha\sin x + \beta\cos x} dx$.

[Hint. Assume

$$a\sin x + b\cos x \equiv A(\alpha\sin x + \beta\cos x) + B(\alpha\cos x - \beta\sin x)$$

and determine A and B by equating like terms. Treat Ex. 23 in like manner.]

23. $\int \frac{ae^x + be^{-x}}{\alpha e^x + \beta e^{-x}} dx$.

24. $\int \frac{\sin(x+a)}{\sin(x+b)} dx$.

CHAPTER VI

INTEGRATION AS A SUMMATION. AREAS

137. Areas. The problem of calculating the area bounded by given straight or curved lines can be solved by means of the Integral Calculus provided that the equations of the boundary curves are known and satisfy certain restrictions.

Suppose it is required to determine the area limited by a continuous arc of a curve whose equation, in rectangular coördinates, is written in the form

$$y = f(x), \qquad (1)$$

by the two ordinates $x = a$ and $x = b$, and by the x-axis; that is, the area $APQB$ (Fig. 59).

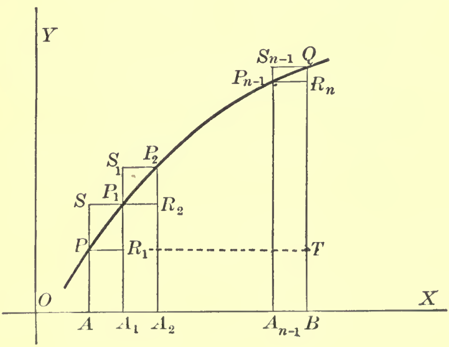

FIG. 59

We proceed as follows. It is assumed in the first place, for the sake of simplicity, that $f(x)$ is always increasing (or always decreasing) between $x = a$ and $x = b$, so that a variable point on the arc PQ is continually rising (or falling) as its abscissa x increases. Suppose, further, that every ordinate between $x = a$ and $x = b$ cuts the arc PQ in but one point. Let the interval A to B (Fig. 59) be divided into n equal intervals AA_1, $A_1A_2 \cdots$, $A_{n-1}B$, each of length Δx, so that

$$\text{interval } AB = b - a = n \cdot \Delta x.$$

At each of the points of division A, A_1, A_2, $\cdots$, B erect ordinates and suppose that these meet the curve in the points P, P_1, P_2, $\cdots$, Q. Through the latter points draw lines PR_1, P_1R_2, P_2R_3, $\cdots$ $P_{n-1}R_n$ parallel to the x-axis.

A series of rectangles PA_1, P_1A_2, $\cdots$ is thus formed, each of which lies entirely within the given area. These will be referred to as the *interior rectangles*. By producing the lines already drawn, a series of rectangles SA_1, S_1A_2, $\cdots$ is formed which will be called the *exterior rectangles*. It is clear that the given area will always be greater than the sum of the interior rectangles and always less than the sum of the exterior, or, expressed in a formula,

$$PA_1 + P_1A_2 + \cdots + P_{n-1}B < \text{Area } APQB < SA_1 + S_1A_2 + \cdots + S_{n-1}B. \tag{2}$$

The difference between the sum of the exterior and the sum of the interior rectangles is

$$SR_1 + S_1R_2 + \cdots + S_{n-1}R_n = \text{rectangle } S_{n-1}T = TQ \cdot \Delta x. \tag{3}$$

As we suppose the curve to be continuous between P and Q, the line TQ is of finite length.

If the number n of equal parts into which AB is divided is increased, the first sum in (2) increases in value and the second sum in (2) decreases. Moreover, as their difference $TQ \cdot \Delta x$, given in (3), approaches the limit zero, it follows that *the limit of the sum of the exterior rectangles is equal to the limit of the sum of the interior rectangles when $n = \infty$, that is, when $\Delta x = 0$.*

Since the required area always has a value intermediate between the two sums, it follows that *the area is equal to the limit of either sum.* So that, for example, we have

$$\text{area} = \lim_{\Delta x \doteq 0} [PA_1 + P_1A_2 + \cdots + P_{n-1}B]. \qquad (4)$$

The second member of this equation may be expressed in terms of the function $f(x)$ which appears in the equation (1) of the given curve. For,

$$\text{area} \quad PA_1 = AP \cdot \Delta x = f(a)\Delta x,$$

since AP is the ordinate y when $x = a$.

Similarly,

$$\text{area} \quad P_1A_2 = A_1P_1 \cdot \Delta x = f(a + \Delta x) \cdot \Delta x,$$

$$\text{area} \quad P_2A_3 = A_2P_2 \cdot \Delta x = f(a + 2\,\Delta x) \cdot \Delta x,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\text{area} \ P_{n-1}B = A_{n-1}P_{n-1} \cdot \Delta x = f(a + \overline{n-1}\,\Delta x) \cdot \Delta x.$$

If these expressions are substituted in (4), it takes the form

$$\text{area} = \lim_{\Delta x \doteq 0} [f(a) + f(a + \Delta x) + f(a + 2\,\Delta x) + \cdots$$

$$+ f(a + \overline{n-1}\,\Delta x)]\Delta x. \qquad (5)$$

As it now stands, the formula just derived is of little practical value for computing areas. This is due to the fact that there is no general method for calculating the sum of the n terms given in brackets in the second member of (5).

Fortunately, the value of the limit of this sum when $n \doteq \infty$ and $\Delta x \doteq 0$ can be calculated by integration as we shall now proceed to show.

138. Expression of area as a definite integral. Denote the function arising from the integration of $f(x)$ by $F(x)$, that is, let $$F(x) = \int f(x)\,dx,$$

or $$\frac{dF(x)}{dx} = f(x).$$

By definition of the derivative of $F(x)$ we have

$$\lim_{\Delta x \doteq 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x).$$

The quotient $\dfrac{F(x + \Delta x) - F(x)}{\Delta x}$ may be written in the form $f(x) + \phi$, in which ϕ approaches zero at the same time as Δx, otherwise the limit of the quotient when $\Delta x = 0$ could not be $f(x)$. From this relation follows, on multiplying by Δx,

$$F(x + \Delta x) - F(x) = f(x) \cdot \Delta x + \phi \cdot \Delta x. \tag{6}$$

Next, in equation (6) substitute for x the successive values

$$a,\ a + \Delta x,\ a + 2\,\Delta x,\ \cdots,\ a + (n-1)\Delta x.$$

We thus deduce the following series of n equations, in which $\phi_1, \phi_2, \cdots$ are used to denote the different values which ϕ may take:

$$F(a + \Delta x) - F(a) = f(a) \cdot \Delta x + \phi_1 \cdot \Delta x,$$

$$F(a + 2\,\Delta x) - F(a + \Delta x) = f(a + \Delta x) \cdot \Delta x + \phi_2 \cdot \Delta x,$$

$$F(a + 3\,\Delta x) - F(a + 2\,\Delta x) = f(a + 2\,\Delta x) \cdot \Delta x + \phi_3 \cdot \Delta x,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F(a + \overline{n-1} \cdot \Delta x) - F(a + \overline{n-2} \cdot \Delta x) = f(a + \overline{n-2} \cdot \Delta x)\Delta x + \phi_{n-1} \cdot \Delta x,$$

$$F(a + n\,\Delta x) - F(a + \overline{n-1} \cdot \Delta x) = f(a + \overline{n-1} \cdot \Delta x)\Delta x + \phi_n \cdot \Delta x.$$

Let these n equations be added; then all but two of the terms in the left member of the sum cancel each other and the result may be written

$$F(b)-F(a)=[f(a)+f(a+\Delta x)+\cdots+f(a+\overline{n-1}\cdot\Delta x)]\Delta x$$
$$+[\phi_1+\phi_2+\cdots+\phi_n]\Delta x,$$

in which b is written for $a+n\,\Delta x$, since $n\,\Delta x=b-a$.

Now let Δx approach zero. The expression

$$(\phi_1+\phi_2+\cdots+\phi_n)\Delta x$$

vanishes at the limit. For, let Φ denote the positive value of the numerically largest term of the set $\phi_1, \phi_2, \cdots, \phi_n$; then we have evidently

$$|(\phi_1+\phi_2+\cdots+\phi_n)\,\Delta x| \leqq (\Phi+\Phi\cdots(n \text{ terms}))\;\Delta x = n\Phi\cdot\Delta x$$
$$= n\Delta x\cdot\Phi=(b-a)\cdot\Phi.$$

Hence, from the fact that $\lim\limits_{\Delta x \doteq 0}\Phi=0$ and that $b-a$ is finite, it follows from Art. 3 that

$$\lim_{\Delta x\doteq 0}(\phi_1+\phi_2+\cdots\phi_n)\,\Delta x=0;$$

and therefore $\quad F(b)-F(a)=\lim\limits_{\Delta x\doteq 0}[f(a)+f(a+\Delta x)+\cdots$

$$+f(a+\overline{n-1}\cdot\Delta x)]\Delta x. \tag{7}$$

Now the right member of this equation is exactly the expression previously derived for the area $APQB$; hence,

$$\text{area } APQB = F(b)-F(a). \tag{8}$$

To compute the value of the right member of (8), first obtain $F(x)$ by integrating $f(x)\,dx$. Having determined $F(x)$, substitute the values b and a which x takes at the extremities of the arc bounding the given area and then subtract the second from the first. This result may conveniently be represented by the symbol

$$\int_a^b f(x)\,dx,$$

which indicates both the integration to be performed and the substitution of the two limiting values a and b for x. It is called *the definite integral of the function* $f(x)$ *between the limits* a *and* b.

We thus obtain, as a final formula for area,

$$\text{area } APQB = \int_a^b f(x)\,dx. \tag{9}$$

139. Generalization of the area formula. Instead of taking the limit of the sum of the interior (or exterior) rectangles, a more

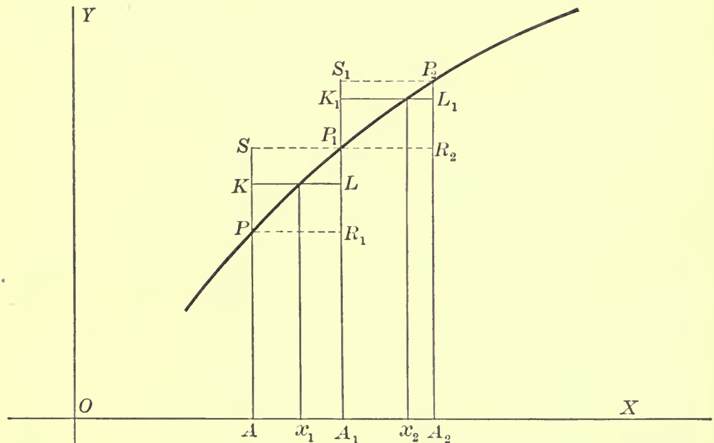

FIG. 60

general procedure would be to take a series of intermediate rectangles. Let x_1 be any value of x between a and $a+\Delta x$, x_2 any value between $a+\Delta x$ and $a+2\,\Delta x$, etc. Then $f(x_1)\,\Delta x$ would be the area of a rectangle KLA_1A (Fig. 60) intermediate between PA_1 and SA_1; that is,

$$PA_1 < f(x_1)\,\Delta x < SA_1.$$

Likewise $$P_1A_2 < f(x_2)\Delta x < S_1A_2, \text{ etc.}$$

Hence,

$$\text{sum of interior rectangles} < [f(x_1) + f(x_2) + \cdots]\Delta x$$
$$< \text{sum of exterior rectangles,}$$

and therefore (cf. Fig. 59),

$$\text{area } APQB = \lim_{\Delta x \doteq 0} [f(x_1) + f(x_2) + \cdots + f(x_n)] \Delta x. \tag{10}$$

This result combined with (9) gives for the definite integral the more general formula:

$$\int_a^b f(x)\,dx = \lim_{\Delta x \doteq 0} [f(x_1) + f(x_2) + \cdots f(x_n)]\Delta x. \tag{11}$$

140. Certain properties of definite integrals. From the definition of the definite integral $\int_a^b f(x)\,dx$ as the limit of a particular sum, certain important properties may be deduced.

(*a*) *Interchanging the limits a and b merely changes the sign of the definite integral.*

For, if x starts at the upper limit b and diminishes by the addition of successive negative increments $(-\Delta x)$, a change of sign will occur in formula (7), giving

$$F(a) - F(b) = \int_b^a f(x)\,dx.$$

Hence,

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

(*b*) *If c is a number between a and b, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

(c) *The Mean Value Theorem.*

The area $APQB$ (Fig. 61), which represents the numerical value of the definite integral may be expressed as follows.

Let an ordinate MN be drawn in such a position that

area $PSN =$ area NRQ.

If ξ denotes the value of x corresponding to the point N, then $MN = f(\xi)$, and

area $APQB =$ rectangle $ASRB$
$= MN \cdot AB = f(\xi)(b-a)$.

Hence,

$$\int_a^b f(x)\,dx = f(\xi)(b-a), \quad (12)$$

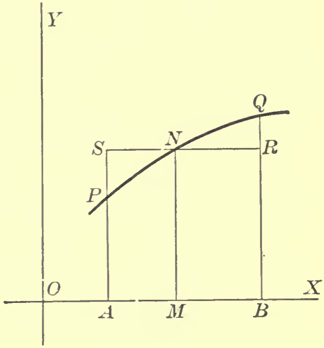

FIG. 61

in which ξ is some value of x between a and b. This result is known as the Mean Value Theorem (compare Art. 39),

and the ordinate $f(\xi) = \dfrac{\int_a^b f(x)\,dx}{b-a}$ is called the *mean ordinate* between $x = a$ and $x = b$. This is also called the mean value of the function $f(x)$ between these limits.

The theorem may be expressed in words as follows:

The value of the definite integral

$$\int_a^b f(x)\,dx$$

is equal to the product of the difference between the limits by the value of the function $f(x)$ corresponding to a certain value $x = \xi$ between the limits of integration.

(d) It is frequently desirable to make a change of variable in the definite integral in order to facilitate the work of integration. It is obvious, from the nature of the definite integral, that the limits of integration must be changed so that in the new integral the limits shall be the values of the new variable corresponding to those of the old variable.

Ex. Evaluate $\int_0^a \sqrt{a^2 - x^2}\, dx.$

Make the change of variable $x = a \sin \theta$, whence $dx = a \cos \theta\, d\theta$, and therefore

$$\int_0^a \sqrt{a^2 - x^2}\, dx = a^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta\, d\theta.$$

Here the limits for the new integral are determined by inspection of the equation connecting x and θ, namely, $\sin \theta = \frac{x}{a}$. It is seen that, as x varies from 0 to a, $\sin \theta$ varies from 0 to 1. This corresponds to a variation of θ between the limits 0 and $\frac{\pi}{2}$. The indefinite integral is, by Art. 134 (c),

$$a^2 \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right).$$

The substitution of the limits gives the value $\frac{a^2 \pi}{4}$.

141. Maclaurin's formula. As an application of the mean value theorem (Art. 140 (c)), we derive Maclaurin's formula with the remainder term.

Let s and t be independent variables. Suppose $f(s - t)$, together with its first n derivatives with respect to t, to be continuous within the interval 0 to t_1. Then we have by integration

$$\int_0^{t_1} f'(s - t)\, dt = -f(s - t)\Big]_0^{t_1} = f(s) - f(s - t_1).$$

On the other hand if we integrate by parts, taking $u = f'(s - t)$, $dv = dt$, we obtain

$$\int_0^{t_1} f'(s - t) dt = f'(s - t) \cdot t\Big]_0^{t_1} + \int_0^{t_1} f''(s - t) \cdot t\, dt$$

$$= f'(s - t_1) \cdot t_1 + \int_0^{t_1} f''(s - t) \cdot t\, dt.$$

Integrate the last term by parts, taking $u = f''(s-t)$, $dv = t\,dt$. By successive applications of this process we deduce the formula

$$f(s)-f(s-t_1)=f'(s-t_1)t_1+f''(s-t_1)\frac{t_1^2}{2!}+f'''(s-t_1)\frac{t_1^3}{3!}+\cdots$$

$$+\frac{1}{(n-1)!}\int_0^{t_1} f^{(n)}(s-t)\,t^{n-1}\,dt.$$

By the mean value theorem we have

$$\int_0^{t_1} f^{(n)}(s-t)\,t^{n-1}\,dt = f^{(n)}(s-\theta t_1)(\theta t_1)^{n-1}\cdot t_1,$$

in which θ is a positive fraction and θt_1 is the same as ξ of (12). Inserting this in the preceding equation and substituting $s=x$, $t_1=x-a$ (hence $s-t_1=a$) we obtain as a final form

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots$$

$$+\frac{\theta^{n-1}}{(n-1)!}f^{(n)}(x-\theta(x-a))(x-a)^n.$$

If we replace θ by $1-\theta'$, the remainder term takes the form given on p. 153, with θ' written in the place of θ.

142. Remarks on the area formula. (*a*) It is noticed that the formula

$$\int_a^b f(x)\,dx = \lim_{\Delta x \doteq 0}[f(a)+f(a+\Delta x)+\cdots+f(a+\overline{n-1}\cdot\Delta x)]\Delta x$$

indicates two steps, — a summation, and a process of passing to a limit. The differential $f(x)\,dx$ which appears under the integral sign may be regarded as representing the general term $f(x)\,\Delta x$ of the series to be summed, while the process of taking the limit of this sum is indicated by replacing Δx with the differential dx and prefixing the sign of integration.

The general term $f(x)\,\Delta x$ represents the area of an arbitrary rectangle (of the set of interior rectangles) whose altitude is the ordinate corresponding to an arbitrary x and whose width is Δx. This is called an *element of area.* The definite integral may then be thought of as indicating the limit of the sum of all contiguous elements of area between $x = a$ and $x = b$.

This notion of summation (followed by passing to the limit $\Delta x = 0$) is a very useful one in applying the calculus to problems of geometry, mechanics, and physics. In each case an application of this notion consists in finding the general expression for an element of the given magnitude (element of area, element of mass, element of moment of inertia, etc.) and then indicating the two steps of summation and taking the limit by changing Δx to dx and prefixing the symbol* of the definite integral. It must not be forgotten that in every case it is necessary to prove that the limit of the sum gives precisely the desired result.† This we have already done in case of the area formula.

(*b*) The element of area $f(x) \cdot \Delta x$ is positive when the corresponding rectangle is above the x-axis, since in that case $f(x)$ is positive, while Δx is positive if $b > a$. Accordingly, *the formula* $\int_a^b f(x)\,dx$ *gives a positive value for an area above the x-axis provided we take* $b > a$.

Similar considerations show that the same formula gives a negative value for an area below the x-axis.

(*c*) If the curve $y = f(x)$ crosses the x-axis between the two points A, B, then the area consists of a positive part APC,

* This symbol originated historically from the initial of the word *sum*.

† In some cases the limit of the sum is used as a definition of the magnitude in question, as, for example, in the definition of the length of arc. (Art. 151.)

represented by the integral $\int_a^c f(x)\,dx$, and a negative part CBQ represented by the integral $\int_c^b f(x)\,dx$. The sum of these two integrals, which (by Art. 140 b) is equal to $\int_a^b f(x)\,dx$, would accordingly give the algebraic sum of the positive and the negative area.

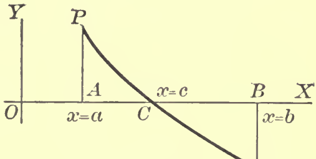

FIG. 62

(d) Some of the restrictions placed upon the function $f(x)$ in Art. 137 can be removed. In the first place, suppose that $f(x)$ is not always increasing (or decreasing) as x increases from a to b. Let ordinates be drawn at the maximum and minimum points of the given arc PQ (Fig. 63). These divide the required area into several parts A', A'', A''' for each of which the ordinates satisfy the original condition of Art. 137, hence we conclude that

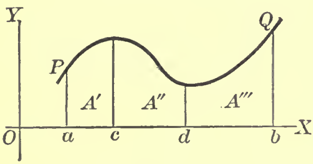

FIG. 63

$$\text{area} = A' + A'' + A''' = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^b f(x)\,dx$$

$$= \int_a^b f(x)\,dx, \text{ by Art. 140 } (b).$$

A discussion of the methods to be employed in case $f(x)$ becomes discontinuous, or is not singly valued in the assigned interval, is postponed to Art. 143.

(e) Since $f(x) = y$, formula (9) may be written more briefly

$$\text{area } APQB = \int_a^b y\,dx. \qquad (13)$$

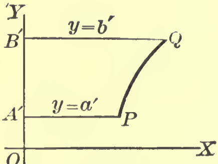

FIG. 64

(*f*) By exactly the same process used in deriving (9), or (13), it may be shown that the area $A'PQB'$ (Fig. 64) bounded by the curve PQ, the y-axis, and the two lines $y = a'$, $y = b'$ is given by the formula

$$\text{area } A'PQB' = \int_{a'}^{b'} x\, dy.$$

(*g*) If it is required to find the area bounded by several arcs such as PQ, QR, RS, etc. (Fig. 65), we may calculate by formula (9) the simple areas $APQB$, $BQRC$, etc., and by proper additions and subtractions obtain the desired area. Thus the area in Fig. 65 would be expressed by

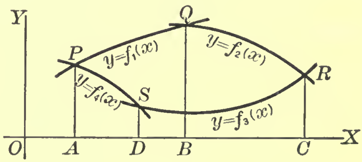

FIG. 65

$$\int_a^b f_1(x)\, dx + \int_b^c f_2(x)\, dx - \int_d^c f_3(x)\, dx - \int_a^d f_4(x)\, dx.$$

EXERCISES

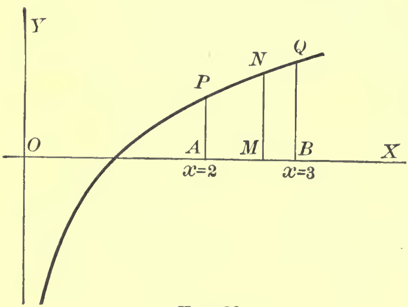

FIG. 66

1. Find the area bounded by the curve $y = \log x$, the x-axis, and the ordinates $x = 2$, $x = 3$.

Area $APQB$ (Fig. 66) $=$

$$\int_2^3 \log x\, dx = x(\log x - 1)\Big]_2^3 {}^{*}$$

$$= 3(\log 3 - 1) - 2(\log 2 - 1)$$

$$= \log \tfrac{27}{4} - 1.$$

* The symbol $\Big]_2^3$ indicates that the values 3 and 2 are to be substituted for x in the expression which precedes the symbol and the second result subtracted from the first.

2. Find the area bounded by the arc of the parabola $y^2 = 4px$ measured from the vertex to the point whose abscissa is a, the x-axis and the ordinate $x = a$.

From the result show that the area of the parabola cut off by a line perpendicular to the axis of the curve is two thirds the area of the rectangle circumscribing this segment.

Does this result hold good for all parabolas?

3. Find the area between the x-axis and one semi-undulation of the curve $y = \sin x$.

4. Find the area bounded by the semicubical parabola $y^2 = 25x^3$ and the line $x = 3$.

5. Find the area bounded by the curve $y^2 = 4(x+5)^3$ and the y-axis.

6. Find the area bounded by the cubical parabola $y = x^3$, the y-axis, and the line $y = 1$.

7. Find the area bounded by the curve $x + y^3 = 2$ and the coördinate axes.

8. Find the area bounded by the parabola $y = 2x^2$ and the line $y = 2x$.

9. Find the area bounded by the parabola $y = x^2$ and the two lines $y = x$ and $y = 2x$.

10. Find by integration the area of the circle $x^2 + y^2 = r^2$.

11. Find the area between the curve $y = x(x-1)(x-3)$ and the x-axis.

12. Find the area bounded by the coördinate axes, the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$, and the ordinate $x = x_1$. By increasing x_1 without limit, find the area between the curve and the x-axis.

13. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

14. Find the area included between the hyperbola $xy = 36$ and the line $x + y = 15$.

15. Find the area bounded by the logarithmic curve $y = a^x$, the x-axis, and the two ordinates $x = x_1$, $x = x_2$. Show that the result is proportional to the difference between the ordinates.

16. Find the area between the curve $y = (x^2 - 1)(x^2 - 2)$ and the x-axis.

17. Find the area cut off from the parabola $(x - 1)^2 = y - 1$ by the line $y = x$.

18. Find the area of the oval in the curve $y^2 = (x - a)(x - b)^2$, given $a < b$.

19. Prove that the area of the curve $a^2y^2 = x^3(2a - x)$ is equal to that of a circle of radius a. Draw figures of the two curves (center of the circle at the point $(a, 0)$) and compare.

20. Find the area of the loop of the curve $y^2 = x^4 + x^5$.

21. Given the curve of damped vibrations $y = e^{-x} \sin x$. Show that the areas contained between successive semi-undulations of the curve, and the positive x-axis form a geometrical series of alternately positive and negative terms.

Find the sum of this infinite series and verify that the same result may be obtained by integrating between the limits 0 and ∞.

Find the total area included between the positive x-axis and the curve (changing the negative areas to positive).

22. Find the area bounded by the hyperbola $xy = a^2$, the x-axis, and the two ordinates $x = a$, $x = na$.

From the result obtained, prove that the area contained between an infinite branch of the curve and its asymptote is infinite.

23. Find the area contained between the curves $y^3 = x$ and $x^3 = y$.

24. Take the segment of the equilateral hyperbola $xy = k^2$, between two points P and Q. Show that the area between this arc and the x-axis is the same as that between the same arc and the y-axis.

25. Find the area bounded by the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the coördinate axes.

26. Find the area between the curve $y^2(y^2-2)=x-1$ and the coördinate axes.

27. Find the area common to the two ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

28. Find the area enclosed by the curves $y=\sin x$, $y=\cos x$ between two consecutive intersections.

29. Find the mean ordinate of the curve $y=\tan x$ between the limits $x=0$ and $x=\frac{\pi}{4}$ (see p. 275).

30. Find the mean value of the function $\sin x$ between the limits 0 and $\frac{\pi}{2}$; also of the function $e^{-x}\sin x$.

31. Find the area of the loop of the curve

$$y^2=x^2\frac{a-x}{a+x}.$$

143. Precautions to be observed in evaluating definite integrals. The method given above for determining plane areas in rectangular coördinates involves two essential steps:

(1) To find the integral of the given function $f(x)$;

(2) To substitute for x the two limiting values a and b, and subtract the first result from the second.

Erroneous conclusions may be reached, however, by an incautious application of this process. The case requiring particular attention is that in which $f(x)$ becomes infinite for some value of x between a and b, or at a or b. When that happens, a special investigation must be made. The method of procedure will be brought out in the following examples.

Ex. 1. Find the area bounded by the curve $y(x-1)^2=c$, the coördinate axes, and the ordinate $x=2$.

A direct application of the formula gives

$$\text{area} = \int_0^2 \frac{c\,dx}{(x-1)^2} = -\frac{c}{x-1}\Big]_0^2 = -2c,$$

where the symbol $\Big]_a^b$ is a sign of substitution, indicating that the values b, a are to be inserted for x in the expression immediately preceding the sign, and the second result subtracted from the first.

This result is incorrect. A glance at the equation of the curve shows that $f(x)\left[= \frac{c}{(x-1)^2}\right]$ becomes infinite for $x = 1$. It is

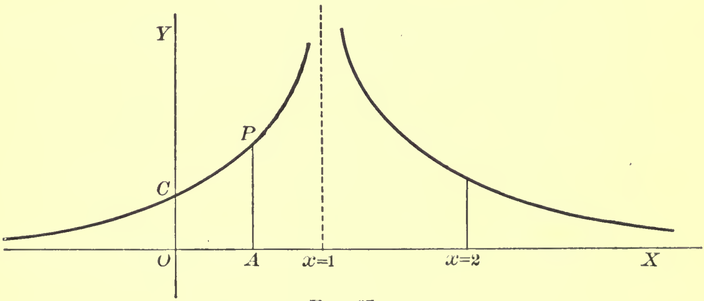

FIG. 67

accordingly necessary to find the area $OCPA$ (Fig. 67) bounded by an ordinate AP corresponding to a value $x = x'$, which is less than 1. For this part of the area $f(x)$ is finite and positive, and formula (9) can be immediately applied, with the result

$$\text{area } OCPA = \int_0^{x'} \frac{c\,dx}{(x-1)^2} = -\frac{c}{(x-1)}\Big]_0^{x'} = -\frac{c}{x'-1} - c. \quad 0 < x' < 1.$$

If now x' is made to increase and approach 1 as a limit, the value of the expression for the area will increase without limit.

A like result is obtained for the area included between the ordinates $x = 1$ and $x = 2$. Hence the required area is infinite.

Ex. **2.** Find the area limited by the curve $y^3(x^2 - a^2)^2 = 8x^3$, the coördinate axes, and the ordinate $x = 3a$.

Since $f(x)\left[=\dfrac{2x}{(x^2-a^2)^{\frac{2}{3}}}\right]$ becomes infinite for $x=a$, it is necessary in the first place to consider the area OPA (Fig. 68) and determine

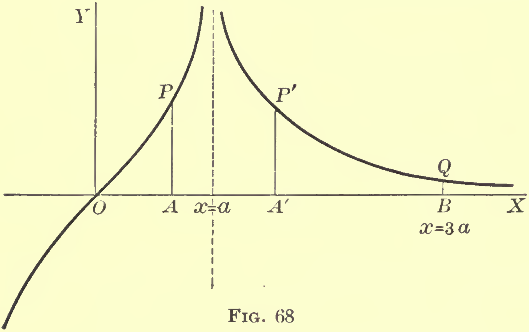

FIG. 68

what limit it approaches as AP approaches coincidence with the ordinate $x=a$. Accordingly

$$\text{area } OPA = \int_0^{x'}\frac{2x\,dx}{(x^2-a^2)^{\frac{2}{3}}} = 3(x^2-a^2)^{\frac{1}{3}}\Big]_0^{x'}$$

$$= 3(x'^2-a^2)^{\frac{1}{3}} + 3a^{\frac{2}{3}}, \qquad 0 < x' < a$$

whence

$$\lim_{x' \doteq a}[\text{area } OPA] = 3a^{\frac{2}{3}}.$$

In the same manner, the area $A'P'QB$ has the value

$$\int_{x'}^{3a}\frac{2x\,dx}{(x^2-a^2)^{\frac{2}{3}}} = 6a^{\frac{2}{3}} - 3(x'^2-a^2)^{\frac{1}{3}}, \qquad a < x' < 3a.$$

As x' diminishes towards a, the area increases to the limiting value $6a^{\frac{2}{3}}$. Hence, by adding the two results, the required area is found to be

$$3a^{\frac{2}{3}} + 6a^{\frac{2}{3}} = 9a^{\frac{2}{3}}.$$

The same result is found by a direct application of (9), viz.:

$$\int_0^{3a}\frac{2x\,dx}{(x^2-a^2)^{\frac{2}{3}}} = 3(x^2-a^2)^{\frac{1}{3}}\Big]_0^{3a} = 9a^{\frac{2}{3}},$$

so that in this case an immediate use of the area formula gives the correct result.

Some of the details in such problems as the two preceding may be omitted. It is unnecessary first to put $x = x'$, a value less than the critical one, and, after integration and substitution of limits, to let x' approach the critical value as a limit. For this is clearly equivalent to taking the critical value at once as the upper limit for the portion of the area to the left of the infinite ordinate (or as the lower limit for the area to the right of this ordinate).

Thus, in case of an infinite ordinate, the rule of procedure becomes:

Calculate separately, by formula (9), *the two portions of area on each side of the infinite ordinate and add the two results.* If one of these portions is infinite, it is not necessary to calculate the other; the required area is infinite.

The formula (9) for area has been deduced under the assumption that the limits a and b are finite. It may happen, however, that the curve $y = f(x)$ approaches the x-axis as an asymptote. It might then be required to determine the strip of area extending to infinity between the curve and its asymptote. The method of procedure for such a case will be explained in the following example.

Ex. **3**. Find the area bounded by the curve $y(x^2 + 1) = 1$ and the x-axis.

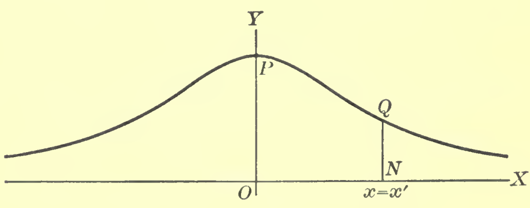

FIG. 69

This curve being symmetrical with respect to the y-axis, it is sufficient to calculate the area in the first quadrant. As our formula of integration does not take account of the case $b = \infty$, we integrate from 0 to x' and in the result cause x' to increase without limit. *This limit will be defined to mean the area between the arc in the positive quadrant, its asymptote, and the y-axis.* It is evident that these steps in the evaluation amount to a direct application of the area formula, using the limits 0 and ∞. The half area is, accordingly,

$$\int_0^\infty \frac{dx}{1+x^2} = \tan^{-1}x\Big]_0^\infty = \tan^{-1}\infty - \tan^{-1}0.$$

We are here confronted with the difficulty that the anti-tangent is a many-valued function and there is a question as to which of its values should be chosen. It is necessary in such a case to go back and examine the limiting process just explained. The area $OPQN$ is equal to $\tan^{-1}x' - \tan^{-1}0$. If x' approaches zero, this expression should approach zero; and as x' increases continuously the area also increases continuously. Accordingly, whatever value we choose for $\tan^{-1}0$, the limit of $\tan^{-1}x'$ should be the value obtained by a *continuous* increase in this function as x' increases without limit. The simplest value for $\tan^{-1}0$ is 0. If $\tan^{-}x'$ increases continuously from 0, it reaches the limit $\frac{\pi}{2}$ when x' becomes infinite. Hence

$$\lim_{x' \doteq \infty} (\tan^{-1}x' - \tan^{-1}0) = \frac{\pi}{2}.$$

If we choose $\tan^{-1}0 = n\pi$, n any integer, then $\lim_{x' \doteq \infty} \tan^{-1}x' = n\pi + \frac{\pi}{2}$, and the difference gives $\frac{\pi}{2}$, as before.

Ex. **4.** Find the area bounded by the curve $y(x^2+a^2)^2 = x$ and the positive x-axis.

Ex. **5.** Find the area bounded by the curve $y = \tan^{-1}x$, the coördinate axes, and the line $x = 1$.

In this problem we have to deal with a many-valued function of x. In fact, to each value of x corresponds an infinite number of values of

$\tan^{-1} x$. The problem, accordingly, has an indefiniteness, which must be removed by making some additional assumption.

The curve $y = \tan^{-1} x$ consists of an infinite number of branches, corresponding ordinates of which differ by integer multiples of π.

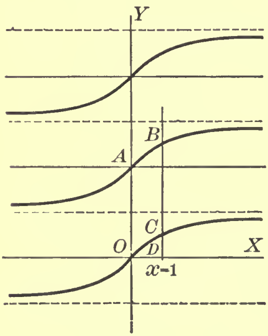

FIG. 70

Each branch is continuous for all finite values of x (see Fig. 70). It is evidently necessary to select one of these branches for the boundary of the proposed area, and discard all the others. Suppose, for example, the branch AB is selected. The ordinate to this branch has the value π when x is zero, and increases continuously to $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$ as x increases continuously to 1. Hence the required area is

$$\int_0^1 \tan^{-1} x \, dx = \left[x \tan^{-1} x - \tfrac{1}{2} \log(x^2 + 1) \right]_0^1$$

$$= \frac{5\pi}{4} - \frac{1}{2} \log 2.$$

EXERCISES

1. Find the area bounded by the curve $y^2(x-1) = 1$, the asymptote $x = 1$, and the line $x = 2$.

2. Find the area bounded by the curve $y^3(x-1)^4 = 1$ and its asymptote, the x-axis.

3. Find the area bounded by the curve of Ex. 2, the x-axis, and the ordinate $x = 2$.

4. Find the area inclosed by the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptote.

5. Find the area bounded by the curve $a^2x = y(x-a)$, the x-axis, and the asymptote $x = a$.

6. Find the area between the cissoid $y^2 = \dfrac{x^3}{2a - x}$ and its asymptote $x = 2a$.

7. Find the area between the curve $y^2(1 - x^2) = 1$ and its asymptotes.

144. Calculation of area when x and y are expressible in terms of a third variable. When the rectangular coördinates of any point of the boundary arc of the required area are given as functions of a third variable θ, we may substitute in $\int_a^b y\,dx$ the expressions for y and dx in terms of θ and integrate between the corresponding new limits for θ in accordance with Art. 140 (*d*).

Area of the cycloid. This curve is traced by a point P in the circumference of a circle of radius r as the circle rolls on a straight line, without sliding.

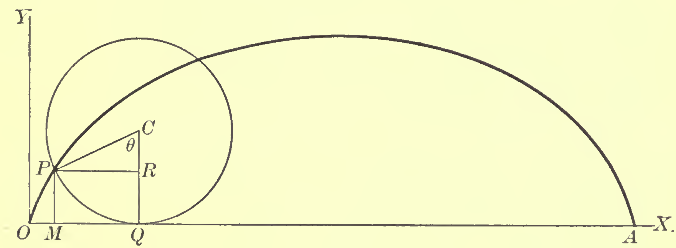

FIG. 71

Let the point P be in contact with the given line at O when the circle begins to roll. Suppose that an arbitrary arc PQ has rolled over the segment OQ. Let (x, y) denote the rectangular coördinates of P, and let θ represent, in radian measure, the angle at the center C subtending PQ; then,

$$OQ = \text{arc } PQ = r\theta.$$

Dropping a perpendicular PR on the line CQ, we have

$$PR = r \sin \theta, \quad RC = r \cos \theta.$$

Accordingly,

$$x = OM = OQ - MQ = r\theta - r \sin \theta = r(\theta - \sin \theta),$$

$$y = MP = QC - RC = r - r \cos \theta = r(1 - \cos \theta).$$

These are called the two parametric equations of the cycloid, θ being a varying parameter. One complete arch of the cycloid is generated as θ varies from 0 to 2π, that is, as x varies from 0 to $2\pi r$. The maximum ordinate for this arc occurs at $x = \pi r$, and the arc is symmetrical with respect to this ordinate.

The area inclosed by the arc OPA and the x-axis is

$$\int_0^{2\pi r} y\, dx = \int_0^{2\pi} r(1 - \cos \theta) \cdot r(1 - \cos \theta)\, d\theta = 3\pi r^2.$$

The area is three times that of the rolling circle.

EXERCISES

1. Find the area of the ellipse when x and y are expressed in terms of the eccentric angle, $x = a \cos \phi$, $y = b \sin \phi$.

What is the meaning of the negative sign in the result?

2. Find the area of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ by expressing x and y in the form $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

3. Find the area of the loop of the folium of Descartes

$$x^3 + y^3 - 3xy = 0.$$

This area may be calculated either by expressing x and y in the form

$$x = \frac{3\theta}{\theta^3 + 1}, \quad y = \frac{3\theta^2}{\theta^3 + 1},$$

obtained by putting $y = \theta x$ and solving for x and y, or by transforming to polar coördinates and using the polar formula for area, Art. 145.

4. Find the area within the curve $y^2 = (1 - x^2)^3$ by assuming $x = \cos\theta$, $y = \sin^3\theta$.

5. Find the area of $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$, the evolute of the ellipse. (See Fig. 51, p. 190.) Express x and y in the form,

$$ax = (a^2 - b^2)\sin^3\theta,\ by = (a^2 - b^2)\cos^3\theta.$$

145. Areas in polar coördinates. Let PQ be an arc of a curve whose equation is given in polar coördinates (ρ, θ). It is required to find the area bounded by this curve and the two assigned radii QP and OQ.

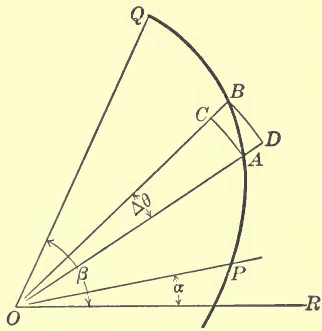

FIG. 72

Let A and B be any two points of the curve with coördinates (ρ, θ) and $(\rho + \Delta\rho,\ \theta + \Delta\theta)$ respectively. Through A draw an arc AC of a circle with radius ρ and center O. The element of area OAC is a sector of a circle of angle $\Delta\theta$. The arc AC is, therefore, $\rho\,\Delta\theta$ and the sectorial area is $\frac{1}{2}\rho^2\,\Delta\theta$. The limit of the sum of all such elements contained between OP and OQ is

$$\frac{1}{2}\int_\alpha^\beta \rho^2\,d\theta. \tag{14}$$

That this is the actual area sought remains to be proved by showing that the sum of the elements of area has the required area for its limit. This may be done by steps exactly analogous to those used in Art. 137, which would consist in proving that the sum of all *interior* sectors, such as OAC, has the same limit as the sum of all *exterior* sectors, such as ODB. The details are left to the student as an exercise.

EXERCISES

1. Find the area of the three loops of the curve $\rho = a \sin 3\theta$.

From the symmetry of the figure it is seen that one sixth of the total area is described as θ varies from 0 to $\frac{\pi}{6}$. Hence the area is

$$6\int_0^{\frac{\pi}{6}} \tfrac{1}{2} a^2 \sin^2 3\theta\, d\theta = \tfrac{3}{2} a^2 \int_0^{\frac{\pi}{6}} (1 - \cos 6\theta) d\theta = \frac{\pi a^2}{4}.$$

This is one fourth the area of the circumscribing circle.

2. Find the area of the lemniscate $\rho^2 = a^2 \cos 2\theta$.

3. Find the area of the circle $\rho = 2r\cos\theta$.

4. Find the area of the cardioid $\rho = r(1 - \cos\theta)$.

5. Find the area of the circle $\rho = 10\sin\theta$.

6. Find the area bounded by the hyperbolic spiral $\rho\theta = c$ and radii drawn to two arbitrary points (ρ_1, θ_1) and (ρ_2, θ_2). Show that the area is proportional to the difference between the radii.

7. Find the area of the four loops of the curve $\rho = a \sin 2\theta$.

8. Find the area of the loop in the spiral of Archimedes $\rho = a\theta$ generated between the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ for θ.

9. Find the area bounded by the lituus $\rho^2\theta = k$ and two arbitrary radii, making angles θ_1 and θ_2 with the polar axis.

10. Find the area of one loop of the curve $\rho^2 = a^2 \cos n\theta$.

11. The radius vector of the logarithmic spiral $\rho = e^{-\theta}$ starts at the angle $\theta = 0$ and rotates positively about the origin an infinite number of times. Determine the area swept over by the radius vector.

12. Find the area of the curve $\rho^4 = \sin^2\theta\cos\theta$.

13. Find the area within the curve $\rho = \cos^2\theta$.

14. Find the area of the innermost loop of the double spiral $\rho = \theta^2$.

146. Approximate integration. The trapezoïdal rule. As shown in Art. 138, the numerical value of the definite integral

$\int_a^b y\,dx$ is the same as that of the area bounded by the curve $y=f(x)$, the x-axis, and the two ordinates $x=a$, $x=b$. When a, b, and the coefficients in $f(x)$ are numerically given, the approximate value of this area, and therefore of the definite integral, can be found by adding the n terms of the series $[f(a)+f(a+\Delta x)+\cdots+f(a+\overline{n-1}\cdot\Delta x)]\Delta x$. The closeness of the approximation improves with increasing values of n. A much more rapid method of approximation is now to be considered.

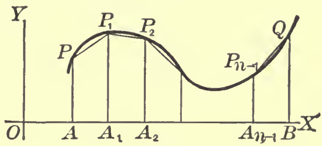

FIG. 73

Instead of forming rectangles, as in Fig. 59, p. 268, draw the chords PP_1, P_1P_2, $\cdots$, $P_{n-1}Q$, thus making trapezoidal elements of area, APP_1A_1, $A_1P_1P_2A_2$, etc. Denote the ordinates at A, A_1, A_2, $\cdots$, A_{n-1}, B by y_0, y_1, y_2, $\cdots$, y_{n-1}, y_n respectively. Also for brevity write $\Delta x = h$. Then the areas of the several trapezoids are

$$APP_1A_1 = \tfrac{1}{2}(y_0 + y_1)h,$$

$$A_1P_1P_2A_2 = \tfrac{1}{2}(y_1 + y_2)h,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{n-1}P_{n-1}QB = \tfrac{1}{2}(y_{n-1} + y_n)h.$$

Hence, by adding, we obtain for the approximate value of the definite integral the expression

$$h\left[\frac{y_0+y_n}{2}+y_1+y_2+\cdots+y_{n-1}\right].$$

This is known as the *trapezoidal formula* for the approximate value of $\int_a^b y\,dx$ and this method of computing its numerical value is called the *trapezoidal rule.*

147. Simpson's rule. *With three ordinates.* Instead of drawing the chords PP_1, P_1P_2 pass a parabola, having its axis vertical, through the three points P, P_1, P_2 and determine the area of the double strip bounded by the two ordinates y_0, y_2, the x-axis, and the parabolic arc.

The equation of the parabola is of the form

$$y = k + lx + mx^2.$$

For convenience take the origin at the foot of the middle ordinate y_1. Then the abscissas of the three ordinates may be represented by $-h$, 0, $+h$, and the area under the parabolic arc is given by the formula

$$\int_{-h}^{h}(k + lx + mx^2)dx = \frac{h}{3}(6k + 2mh^2).$$

This result can be expressed in a simple form in terms of the three ordinates y_0, y_1, y_2. For,

$$y_0 = k - lh + mh^2,$$

$$y_1 = k,$$

$$y_2 = k + lh + mh^2;$$

therefore,
$$y_0 + y_2 = 2k + 2mh^2,$$

hence,
$$6k + 2mh^2 = y_0 + 4y_1 + y_2,$$

and, accordingly,

$$\text{parabolic area } APP_1P_2A_2 = \frac{h}{3}(y_0 + 4y_1 + y_2). \qquad (15)$$

This is Simpson's parabolic formula for three ordinates.

With n ordinates. In like manner the area bounded by the two ordinates y_2, y_4 and a parabolic arc through P_2, P_3, P_4 is

$$\frac{h}{3}(y_2 + 4y_3 + y_4), \qquad (16)$$

and so on. If the number of ordinates $y_0, y_1, \cdots, y_n$ is odd, we obtain, by adding together the expressions (15), (16), etc.

$$\frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n].$$

This is Simpson's formula for the approximate value of $\int_a^b y\,dx$.

148. The limit of error in approximate integration. The approximate value obtained for $\int_a^b f(x)\,dx$ by means of Simpson's formula differs from the true value by an amount which does not exceed*

$$-\frac{(b-a)f^{\text{IV}}(\xi)h^4}{180},$$

in which $f^{\text{IV}}(\xi)$ is the value of the fourth derivative of $f(x)$ when x is given a certain value ξ between a and b. The limit of error for the trapezoidal rule is*

$$-\frac{(b-a)f''(\xi)h^2}{12}.$$

Since ξ is not definitely known, in applying the above formulas to find the limit of error it is necessary to choose ξ so that $f^{\text{IV}}(\xi)$ or $f''(\xi)$ has its greatest value in the interval from a to b. The result so obtained may be considerably larger than would be given by the formula if ξ were actually known. In some cases the result will be so large as to give no useful information in regard to the closeness of our approximation. In other cases it will be small enough to indicate that the required degree of approximation has been attained.

For example, suppose it is required to evaluate

$$\int_{20}^{30}\frac{\log_{10} x}{x}\,dx.$$

* See Markoff, "Differenzenrechnung," § 14, pp. 57, 59.

Since $f(x) = x^{-1}\log_{10}x$, we obtain by successive differentiation $f^{\text{IV}}(x) = x^{-5}(24\log_{10}x - 50 \cdot M)$, $M = \log_{10}e = 0.4343$, very nearly. As we cannot readily determine by inspection the largest numerical value of $f^{\text{IV}}(x)$ in the interval $20 \leqq x \leqq 30$, we obtain the next derivative

$$f^{\text{V}}(x) = x^{-6}(274\,M - 120\log_{10}x).$$

The first factor x^{-6} is positive. The second factor takes a negative value for $x \geqq 20$ and hence $f^{\text{V}}(x)$ is negative in the given interval. Therefore, $f^{\text{IV}}(x)$ is a decreasing function for all the values of x under consideration. But $f^{\text{IV}}(x)$ is positive for $x \doteq 30$, and accordingly its greatest numerical value occurs for $x = 20$, which is $f^{\text{IV}}(20) = 0.000003$.

The limit of error for Simpson's formula is, therefore,

$$-\frac{10(0.000003)}{180}h^4 = -(0.0000002)h^4.$$

If we use 3 ordinates, then $h = 5$ and the error does not exceed -0.0001^+; that is, the error is less than two units in the fourth place of decimals.

EXERCISES

In the following problems use Simpson's formula whenever an odd number of ordinates is given. Determine the limit of error and, when possible by direct integration, the exact error. Also evaluate by using the trapezoidal rule, and compare the degree of accuracy attained by the two different methods.

1. Evaluate $\int_0^4 x^2\,dx$ by the trapezoidal rule, using 5 ordinates; 9 ordinates.

In the case of 9 ordinates, $n = 8$ and $h = \dfrac{b-a}{n} = \dfrac{1}{2}$, $y_0 = 0$, $y_1 = (\frac{1}{2})^2$, $y_2 = 1$, $y_3 = (\frac{3}{2})^2, \cdots, y_8 = 4^2$.

2. Prove that Simpson's rule gives the exact value of $\int_a^b x^2\,dx$, $\int_a^b x^3\,dx$, $\int_a^b (\alpha x^3 + \beta x^2 + \gamma x + \delta)\,dx$.

3. Evaluate $\int_0^{\frac{\pi}{6}} \cos x\,dx$, using 3 ordinates; 5 ordinates; 7 ordinates; 9 ordinates. Notice the variation of error with increasing values of n.

4. Evaluate $\int_0^4 \sqrt{x}\,dx$, using 5 ordinates.

5. Evaluate $\int_0^6 \sqrt{1 + x^3}\,dx$, using 4 ordinates; 7 ordinates.

6. Evaluate $\int_0^{\frac{\pi}{2}} \cos x\,dx$, using 7 ordinates.

7. Evaluate $\int_2^{12} \log_{10} x\,dx$, using unit intervals.

8. Evaluate $\int_{10}^{70} \frac{dx}{\log_{10} x}$, using 7 ordinates.

9. Evaluate $\int_0^1 \sqrt{1 - x^4}\,dx$, using 6 ordinates.

10. Evaluate $\int_0^1 e^{-x_2}\,dx$, using 11 ordinates.

This integral (with any upper limit) is called the *Probability Integral* since it plays an important rôle in the theory of probabilities.

11. Evaluate $\int_0^{\frac{\pi}{6}} \sqrt{1 - 3\sin^2 x}\,dx$, using 7 ordinates.

12. Evaluate $\int_0^{10} x^2\,dx$ by the trapezoidal rule, using 11 ordinates.

13. Calculate the value of π from the formula $\frac{\pi}{4} = \int_0^1 \frac{dx}{1 + x^2}$, using 11 ordinates.

Determine the error by comparison with the known value of π.

14. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,d\theta$, taking θ at intervals 15°, 10°, 9°.

This, like Ex. 11, is an *Elliptic Integral* and cannot be integrated by any formula given in the present volume. It occurs in the problem of calculating friction in journals. (See "Engineering Mathematics" by Prof. V. Karapetoff, Part I, p. 16. Wiley, 1912.)

15. Evaluate $\int_{20}^{30} \frac{\log_{10} x}{x}\,dx$, using 3 ordinates.

CHAPTER VII

GEOMETRICAL APPLICATIONS

149. Volumes by single integration. The volumes of various solids may easily be calculated by a summation process exactly similar to that used in computing areas. The following problems will make the mode of procedure clear.

Ex. 1. A woodman fells a tree 2 ft. in diameter, cutting halfway through on each side. The lower face of each cut is horizontal and the upper face makes an angle of 60° with the lower. How much wood does he cut out?

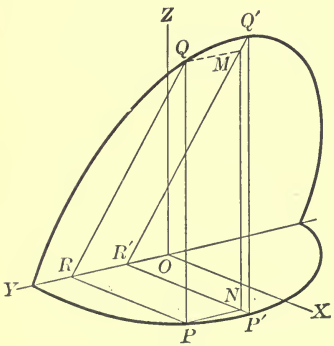

Fig. 74

The portion cut out on one side forms a solid bounded by a cylindrical surface whose equation may be taken in the form $x^2 + y^2 = 1$, and by two planes whose intersection may be chosen for the y-axis. Imagine this wedge-shaped solid divided into thin plates by means of planes parallel to the xz-plane and at equal distances Δy. The volume of an arbitrary plate $PQRP'Q'R'$ is approximately equal to the area of the triangular face multiplied by the thickness Δy.

$$\text{Area } PQR = \tfrac{1}{2}\, RP \cdot PQ = \tfrac{1}{2}\, xz = \frac{\sqrt{3}}{2}\, x^2,$$

since $\frac{z}{x} = \tan 60° = \sqrt{3}$. The element of volume is therefore

$$\frac{\sqrt{3}\,x^2}{2}\cdot\Delta y.$$

Since the figure is symmetrical with respect to the xz-plane, it is sufficient to calculate the volume between the limits 0 and 1 for y and double the result.

The limit of the sum of all elements of volume in the first octant is

$$\frac{\sqrt{3}}{2}\int_0^1 x^2\,dy = \frac{\sqrt{3}}{2}\int_0^1 (1-y^2)\,dy = \frac{1}{\sqrt{3}}\cdot$$

That this limit is the volume to be determined may be seen on observing that the element of volume falls short of the total amount contained in the plate $PQRP'Q'R'$ by the prismatic piece $PNP'QMQ'$. The sum of all these neglected portions, in the first octant, is less than the volume of the maximum plate (having the xz-plane for base), and hence approaches zero as Δy diminishes.

Therefore the total volume of wood cut out is $\frac{4}{\sqrt{3}}$ cu. ft.

Ex. **2**. Calculate the volume in Ex. 1, by dividing the solid of Fig. 74 with equidistant planes parallel to the yz-plane.

Ex. **3**. Find the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Imagine the solid divided into a number of thin plates by means of planes perpendicular to the x-axis and at equal distances Δx. Regard the volume of each plate as approximately that of an elliptic cylinder of altitude Δx, whose base is the section of the ellipsoid by one of the cutting planes. If the equation of this plane is $x = \lambda$, the equation of the elliptic base of the plate is (in y, z coördinates)

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{\lambda^2}{a^2}.$$

Dividing by $1 - \frac{\lambda^2}{a^2}$, we obtain

$$\frac{y^2}{b^2\left(1 - \frac{\lambda^2}{a^2}\right)} + \frac{z^2}{c^2\left(1 - \frac{\lambda^2}{a^2}\right)} = 1.$$

The semiaxes of the ellipse are

$$b\sqrt{1 - \frac{\lambda^2}{a^2}},\ c\sqrt{1 - \frac{\lambda^2}{a^2}}.$$

Since the area of the ellipse is the product of the semiaxes multiplied by π (Ex. 13, p. 281), it follows that the area of the elliptic base is $\pi bc\left(1 - \frac{\lambda^2}{a^2}\right)$. On replacing λ by x, the element of volume may be written

$$\pi bc\left(1 - \frac{x^2}{a^2}\right)\Delta x.$$

The sum of all such elements for values of x varying by equal increments Δx between 0 and a differs from the volume of the half ellipsoid by a series of ring-shaped portions, the total sum of which is less than the volume of the maximum plate of the figure. It readily follows from this that the total volume of the ellipsoid is

$$2\int_0^a \pi bc\left(1 - \frac{x^2}{a^2}\right)dx = \frac{4}{3}\pi abc.$$

Ex. **4**. Solve Ex. 3 by taking the cutting planes parallel to the xz-plane and at equal distances Δy.

Ex. **5**. Solve Ex. 3 by taking the cutting planes parallel to the xy-plane.

Ex. **6**. Find the volume of the portion of the elliptic paraboloid $\frac{y^2}{a^2} + \frac{z^2}{b^2} = x$ cut off by the plane $x = 1$.

Ex. 7. Find the volume of the elliptic cone $\frac{y^2}{a^2}+\frac{z^2}{b^2}=(x-1)^2$ measured from the yz-plane as base to the vertex (1, 0, 0).

Ex. 8. Find the volume of a pyramid of altitude h and of base area A.

[Hint. Take the base on the xy-plane, the altitude coinciding with the z-axis. Cut the solid into thin plates by planes parallel to the base.]

Ex. 9. Given an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. On the major axis a plane rectangle $ABCD$ is constructed perpendicular to the plane of the ellipse. Through any point P of the line CD a plane is constructed perpendicular to CD. The two points R and S in which the latter plane meets the ellipse are joined to P by straight lines. The totality of all lines so determined forms a ruled surface called a conoid. Given $AC=p$, find the volume of the above conoid.

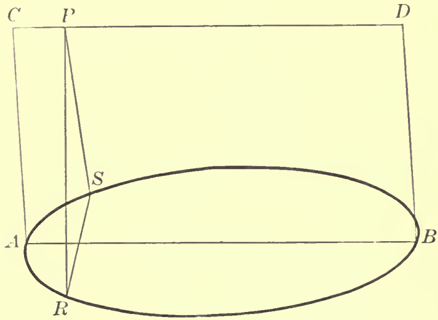

Fig. 75

Ex. 10. A rectangle moves from a fixed point P parallel to itself, one side varying as the distance from P, and the other as the square of this distance. At the distance of 2 ft., the rectangle becomes a square of 3 ft. on each side. What is the volume generated?

Ex. 11. The center of a square moves along a diameter of a given circle of radius a, the plane of the square being perpendicular to that of the circle, and its magnitude varying in such a way that two opposite vertices move on the circumference of the circle. Find the volume of the solid generated.

Ex. **12.** A right circular cone having an angle 2θ at the vertex has its vertex on the surface of a sphere of radius a and its axis passing through the center of the sphere. Find the volume of the portion of the sphere which is exterior to the cone.

Ex. **13.** Find the volume of the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$ cut off by the plane $z=c$.

Ex. **14.** A banister cap is bounded by two equal cylinders of revolution of radius r whose axes intersect at right angles in the plane of the base of the cap. Find the volume of the cap.

150. Volume of solid of revolution. Let the plane area, bounded by an arc PQ of a given curve (referred to rectangular axes) and the ordinates at the extremities P and Q, be revolved about the x-axis. It is required to find the volume of the solid so generated.

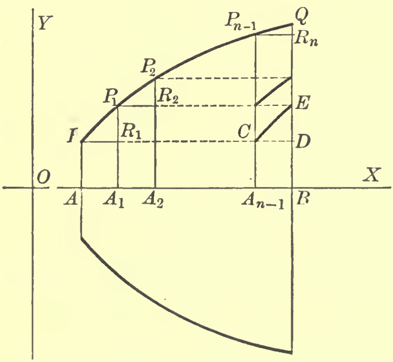

Fig. 76

Let the figure $APQB$ be divided into n strips of width Δx by means of the ordinates A_1P_1, A_2P_2, $\cdots$, $A_{n-1}P_{n-1}$. In revolving about the x-axis, the rectangle APR_1A_1 generates a cylinder of altitude Δx, the area of whose base is $\pi\cdot\overline{AP}^2$. Hence

$$\text{volume of cylinder} = \pi\cdot\overline{AP}^2\cdot\Delta x.$$

The volume of this cylinder is less than that generated by the strip APP_1A_1 by the amount contained in the ring generated by the triangular piece PR_1P_1. Imagine this ring pushed in the direction of the x-axis until it occupies the position of the ring generated by CDE. If every other neglected portion (such as is generated by $P_{i-1}P_iR_i$) is treated in like manner, it is evident that the sum is less than the volume generated by the strip $A_{n-1}P_{n-1}QB$, and hence has zero for limit as Δx approaches zero. Therefore the sum of the n cylinders generated by the interior rectangles of the plane, viz.

$$\pi(\overline{AP}^2 + \overline{A_1P_1}^2 + \cdots + \overline{A_{n-1}P_{n-1}}^2)\Delta x,$$

has for limit the volume required. But the limit of this sum is the definite integral $\int_a^b \pi y^2 dx$, and hence

$$\text{volume} = \pi \int_a^b y^2\, dx.$$

The volume generated by revolution about the y-axis is found by a like process to be expressed by the definite integral

$$\pi \int_{a'}^{b'} x^2\, dy,$$

in which a' and b' are the values of y at the extremities of the given arc.

When the axis of revolution does not coincide with either of the coördinate axes, a similar procedure will usually give at once the element of volume. Examples 1–3 will illustrate.

Ex. Find the volume of revolution of the segment of the parabola $y^2 = x$ cut off by the line $y = x$, the axis of revolution being the given line.

Let OQ be the axis, and P any point of the parabolic arc.

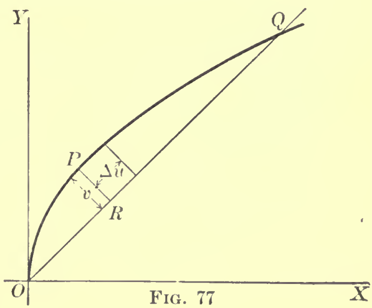

FIG. 77

If v denotes the perpendicular distance PR from P to OQ and u the length of the line OR, then the element of volume is

$$\pi v^2 \, \Delta u.$$

The formula of analytic geometry for the distance from a point to a line gives

$$v = \frac{y - x}{\sqrt{2}} = \frac{\sqrt{x} - x}{\sqrt{2}},$$

in which (x, y) are the coördinates of P. The second form for v is obtained by substituting for y the expression given by the equation of the parabola.

Since Δu is measured on a line making an angle of 45° with the x-axis, it follows that $\Delta u = \sqrt{2} \cdot \Delta x$.

Hence the required volume is

$$\int_0^1 \pi \left(\frac{\sqrt{x} - x}{\sqrt{2}} \right)^2 \sqrt{2} \, dx = \frac{\pi}{30\sqrt{2}}.$$

EXERCISES

1. A quadrant of a circle revolves about its chord. Find the volume of the spindle so generated.

[Hint. Take the equation of the circle in the form $x^2 + y^2 = r^2$ and the equation of the chord $x + y = r$.]

2. Find the volume of revolution of the segment of the circle $x^2 + y^2 = r^2$ cut off by the line $x = a$, this line being the axis of revolution.

3. Find the volume of the truncated cone obtained by revolving about the y-axis the segment of the line $3x + y = 5$ between the points $(2, -1)$ and $(1, 2)$.

4. Find the volume generated by the revolution of the cissoid $y^2 = \dfrac{x^3}{2a - x}$ about the x-axis from the origin to the point (x_1, y_1).

What is the limit of this volume as x_1 approaches $2a$?

5. Find the volume obtained by revolving the entire cissoid about its asymptote, the line $x = 2a$.

[HINT. The element of volume is $\pi(2a - x)^2 \Delta y$. For the purpose of integration express x and y in terms of a third variable t by means of the equations

$$x = 2a \sin^2 t, \quad y = 2a \frac{\sin^3 t}{\cos t}\,]$$

6. Find the volume of the oblate spheroid obtained by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its minor axis.

7. Find the volume of the sphere obtained by revolving the circle $x^2 + (y - k)^2 = r^2$ about the y-axis.

8. The arc of the hyperbola $xy = k^2$, extending from the vertex to infinity is revolved about its asymptote. Find the volume generated.

What is the volume generated by revolving the same arc about the other asymptote?

9. Find the entire volume obtained by rotating the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about either axis.

10. Find the volume obtained by the revolution of that part of the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ intercepted by the coördinate axes about one of those axes.

11. Find the volume generated by the revolution of the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$ about the x-axis.

12. Find the volume generated by the revolution of the witch about the y-axis, taking the portion of the curve from the vertex $(x = 0)$ to the point (x_1, y_1).

What is the limit of this volume as the point (x_1, y_1) moves toward infinity?

13. Find the volume obtained by revolving a complete arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis.

$$\text{Volume} = \pi\int_0^{2\pi a} y^2\,dx = \pi a^3 \int_0^{2\pi}(1 - \cos\theta)^3\,d\theta.$$

14. Find the volume obtained by revolving the cardioid $\rho = a(1 - \cos\theta)$ about the polar axis.

Assume $\qquad x = \rho\cos\theta,\quad y = \rho\sin\theta.$

Then
$$dx = d(\rho\cos\theta) = d[a(1 - \cos\theta)\cos\theta]$$
$$= a\sin\theta(-1 + 2\cos\theta)\,d\theta.$$

Hence
$$\text{volume} = \pi\int y^2\,dx = -\pi a^3\int_0^{\pi}\sin^3\theta(1 - \cos\theta)^2(1 - 2\cos\theta)\,d\theta.$$

151. Lengths of curves. Rectangular coördinates. Let it be required to determine the length of a continuous arc PQ of a curve whose equation is written in rectangular coördinates (x, y).

It is first necessary to define what is meant by the length of a curve. For this purpose, suppose a series of points $P_1, P_2, \cdots, P_{n-1}$ taken on the arc PQ (Fig. 78), and imagine the lengths of the chords $PP_1, P_1P_2, \cdots$ to have been determined. The limit of the sum of these chords as the length of each chord approaches zero will be taken, in accordance with accepted usage, as the definition of the length of the arc PQ;* that is,

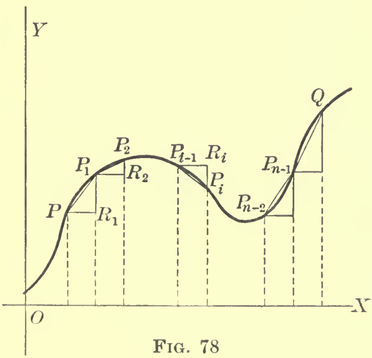

FIG. 78

$$\text{arc } PQ = \text{Lt}\,(\text{chord } PP_1 + \text{chord } P_1P_2 + \cdots + \text{chord } P_{n-1}Q). \quad (1)$$

* That this limit is always the same no matter how the points P_i are chosen, as long as the curve has a continuously turning tangent, and the distances

This definition is immediately convertible into a formula suitable for direct application.

For, let the points P_1, P_2, $\cdots$ be so chosen that

$$PR_1 = P_1R_2 = \cdots = \Delta x,$$

the lines PR_1, etc., being drawn parallel to the x-axis.

Denote by Δy the increment R_1P_1 of y. Then the length of the chord PP_1 is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x = \sqrt{1 + \left(\frac{\Delta x}{\Delta y}\right)^2}\,\Delta y. \tag{2}$$

Now $\frac{\Delta y}{\Delta x}$ is the slope of PP_1. It is, therefore, equal to the slope of that tangent to the arc PP_1 which is parallel to the chord. If (x_1, y_1) denote the coördinates of the point of contact of this tangent line, then we have

$$\frac{\Delta y}{\Delta x} = \frac{dy_1}{dx_1}.$$

Hence the length of chord PP_1 may be expressed in the form $f(x_1)\,\Delta x$, in which

$$f(x_1) = \sqrt{1 + \left(\frac{dy_1}{dx_1}\right)^2}. \tag{3}$$

Similarly

$$P_1P_2 = f(x_2)\,\Delta x, \quad P_2P_3 = f(x_3)\,\Delta x, \ \cdots,$$

in which x_2 is the abscissa of a certain point on the arc P_1P_2, and so for x_3, $\cdots$. When these expressions are substituted in (1), it becomes

$$\text{arc } PQ = \lim_{\Delta x \doteq 0} [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x.$$

$P_{i-1}P_i$ are all made to tend towards zero, admits of rigorous proof. The proof is, however, unsuitable for an elementary textbook. (See Rouché et Comberousse, "Traité de géométrie," Part I, p. 189, Paris, 1891).

But, by (11), p. 274, this limit is $\int_a^b f(x)\,dx$. Substituting for $f(x)$ from (3), we obtain the formula

$$\text{arc } PQ = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx, \tag{4}$$

in which a and b are the abscissas of P and Q, respectively.

Taking for PP_1 the second form in (2), namely,

$$\sqrt{1+\left(\frac{\Delta x}{\Delta y}\right)^2}\,\Delta y,$$

we deduce in like manner

$$\text{arc } PQ = \int_{a'}^{b'} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy,$$

in which a' and b' are the ordinates of P and Q.

EXERCISES

1. Find the length of arc of the parabola $y^2 = 4\,px$ measured from the vertex to one extremity of the latus rectum.

In this case
$$\frac{dy}{dx} = \sqrt{\frac{p}{x}},$$

hence
$$\text{length of arc} = \int_0^p \sqrt{1+\frac{p}{x}}\,dx = \int_0^p \frac{x+p}{\sqrt{x^2+px}}\,dx.$$

2. Find the length of arc of the semicubical parabola $ay^2 = x^3$ from the origin to the point whose abscissa is $\frac{a}{4}$.

3. Find the length of arc of the curve $y = \log\cos x$, measured from the origin to the point whose abscissa is $\frac{\pi}{6}$.

4. Find the entire length of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

5. Find the length of arc of the circle $x^2 + y^2 = r^2$.

6. Find the length of arc of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ from the point $(0, a)$ to the point whose abscissa is a.

7. Find the length of arc of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ between the limits $x = 1$ and $x = 2$.

8. Find the length of the logarithmic curve $y = \log x$ from $x = 1$ to $x = \sqrt{3}$.

9. Find the length of arc of the evolute of the ellipse

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

10. Find the length of arc of the curve $y = a \log (a^2 - x^2)$ from $x = 0$ to $x = \frac{a}{2}$.

152. Lengths of curves. Polar coördinates. The polar formulas for length of arc may be derived from those of the previous article by transformation from rectangular to polar coördinates.

Since $x = \rho \cos \theta$, $y = \rho \sin \theta$, we obtain by differentiating with respect to θ

$$dx = \left(\frac{d\rho}{d\theta}\cos\theta - \rho \sin\theta\right)d\theta, \quad dy = \left(\frac{d\rho}{d\theta}\sin\theta + \rho\cos\theta\right)d\theta,$$

hence

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2}\,d\theta.$$

Therefore the length of arc is

$$\text{arc } PQ = \int_\alpha^\beta \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\,d\theta, \tag{5}$$

the limits of integration being the values of θ at P and Q.

If ρ instead of θ is taken as the independent variable, we deduce in like manner

$$\text{arc } PQ = \int_{\rho_1}^{\rho_2} \sqrt{1 + \left(\rho \frac{d\theta}{d\rho}\right)^2}\, d\rho,$$

the limits being the values of ρ at P and Q.

EXERCISES

1. Find the length of arc of the logarithmic spiral $\rho = e^{\frac{\theta}{a}}$ between the two points (ρ_1, θ_1) and (ρ_2, θ_2), and show that it is proportional to the difference of the two radii ρ_1 and ρ_2.

2. Find the length of arc of the circle $\rho = 2\,a \sin\theta$.

3. Find the entire length of the cardioid $\rho = a(1 - \cos\theta)$.

4. Find the length of the parabola $\rho = a \sec^2 \frac{\theta}{2}$ between the points (ρ_1, θ_1) and (ρ_2, θ_2).

5. Find the length of the spiral of Archimedes $\rho = a\theta$ between two arbitrary points.

6. Find the length of arc of the spiral $\rho = \theta^2$ measured from $\theta = 0$ to $\theta = \pi$.

7. Find the entire length of the curve $\rho = \cos^2\theta$.

8. Find the entire length of the curve $\rho = a \sin^3 \frac{\theta}{3}$.

9. Find the length of arc of the cissoid $\rho = 2\,a \tan\theta \sin\theta$ between the limits 0 and $\frac{\pi}{4}$.

[Hint. For the purpose of integration, express the integrand in terms of $\sec\theta$ as the independent variable.]

153. Measurement of arcs by the aid of parametric representation. Suppose the rectangular coördinates of a point on a given curve are expressed in terms of a third variable t. Then, since in rectangular coördinates $\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ (Art. 41), we have

$$s=\int ds=\int\frac{ds}{dt}\,dt=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt,$$

in which $s=$ arc PQ, and t_1, t_2 are the values of t corresponding to the points P and Q. In like manner, if the polar coördidates (ρ, θ) are expressed in terms of t, the formula for length of arc is

$$s=\int_{t_1}^{t_2}\sqrt{\left(\frac{d\rho}{dt}\right)^2+\left(\rho\frac{d\theta}{dt}\right)^2}\,dt,$$

since
$$\frac{ds}{dt}=\sqrt{\left(\frac{d\rho}{dt}\right)^2+\left(\rho\frac{d\theta}{dt}\right)^2}\cdot \qquad \text{(Art. 45)}$$

EXERCISES

1. Find the length of a complete arch of the cycloid

$$x=a(t-\sin t),\quad y=a(1-\cos t).$$

2. Find the length of the epicycloid

$$x=a(m\cos t-\cos mt),\quad y=a(m\sin t-\sin mt)$$

from $t=0$ to $t=\frac{2\pi}{m-1}\cdot$

3. Find the length of arc of the hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ by expressing x and y in the form $x=a\sin^3 t$, $y=a\cos^3 t$.

4. Find the length of the involute of the circle

$$x=a(\cos t+t\sin t),\quad y=a(\sin t-t\cos t)$$

from $t=0$ to $t=t_1$.

5. Find the length of arc of the curve $x^{\frac{2}{3}}-y^{\frac{2}{3}}=a^{\frac{2}{3}}$ from $(a, 0)$ to (x_1, y_1) by assuming $x=a\sec^3 t$, $y=a\tan^3 t$.

6. Find the length of arc of the curve $x = e^t \sin t$, $y = e^t \cos t$ from $t = 0$ to $t = t_1$.

7. Find the length of arc of the curve $x = a + t^2$, $y = b + t^3$, measured from the point $t = 0$ to the point $t = t_1$.

154. Area of surface of revolution. Let AQ be a continuous arc of a curve whose equation is expressed in rectangular coördinates x and y. It is required to determine a formula for the area of the surface generated by revolving the arc AQ about the x-axis.

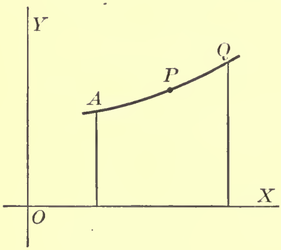

FIG. 79

It has been shown in Art. 44, p. 81, that if S denotes the area of the surface generated by the rotation of AP (P being a variable point with coördinates (x, y)), then ΔS satisfies the conditions of inequality

$$2\,\pi y\,\Delta s < \Delta S < 2\,\pi(y + \Delta y)\,\Delta s. \tag{6}$$

Let the arc AQ be divided into n equal parts of length Δs. For each segment of arc there will be a set of conditions such as (6), the values of y, Δy, ΔS being in general different for the different segments. Let the n sets of inequalities thus obtained be added. In what follows, the symbol $\sum$ will be used as an abbreviation of the expression, "The sum of the n terms of the form." Since $\sum \Delta S = S$ (in which S now denotes the entire surface generated by arc AQ), we have

$$2\,\pi \sum y\,\Delta s < S < 2\,\pi \sum (y + \Delta y)\,\Delta s. \tag{7}$$

Now let Δs (and hence Δy) approach zero. The first member of (7) becomes $2\,\pi \int y\,ds$, which changes to

$$2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx, \text{ or to } 2\pi\int_{a'}^{b'} y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy,$$

on making x, or y, the independent variable. The limit of the last member of (7) may be written

$$\lim_{\Delta s \doteq 0}\sum[y\,\Delta s+\Delta y\,\Delta s]=\int y\,ds+\lim\sum\Delta y\,\Delta s.$$

The last term is zero. For, let δ represent the maximum value of Δy in any of the terms of $\sum\Delta y\,\Delta s$. Then follows

$$\sum\Delta y\,\Delta s \leqq \delta\sum\Delta s=\delta\cdot\text{arc }AQ,$$

and since δ approaches zero, we conclude that $\lim\sum\Delta y\,\Delta s=0.$

Hence $$\lim\sum y\,\Delta s=\lim\sum(y+\Delta y)\,\Delta s,$$

and therefore

$$S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=2\pi\int_{a'}^{b'} y\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy.$$

In like manner the area of the surface obtained by revolving arc AQ about the y-axis is

$$2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=2\pi\int_{a'}^{b'} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy. \qquad (8)$$

EXERCISES

1. Find the surface of the catenoid obtained by revolving the catenary $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$ about the y-axis, from $x=0$ to $x=a$.

Since $$\frac{dy}{dx}=\tfrac{1}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}}),$$

it follows that

$$1+\left(\frac{dy}{dx}\right)^2=\frac{(e^{\frac{x}{a}}+e^{-\frac{x}{a}})^2}{4};$$

hence, by using the first formula of (8), the required surface has the area

$$\pi\int_0^a x(e^{\frac{x}{a}} + e^{-\frac{x}{a}})\,dx.$$

2. Find the surface obtained by revolving about the y-axis the quarter of the circle $x^2 + y^2 + 2x + 2y + 1 = 0$ contained between the points where it touches the coördinate axes.

3. Find the surface generated by revolving the parabola $y^2 = 4px$ about the x-axis from the origin to the point $(p, 2p)$.

4. Find the surface generated by the revolution about the y-axis of the same arc as in Ex. 3.

5. Find the surface generated by the revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

(a) about its major axis (the prolate spheroid);
(b) about its minor axis (the oblate spheroid).

6. Find the surface generated by the revolution of the cardioid $\rho = a(1 + \cos\theta)$ about the polar axis.

Regarding the figure as referred in the first place to rectangular axes such that $x = \rho\cos\theta$, $y = \rho\sin\theta$ we have

$$\text{surface} = 2\pi\int y\,ds = 2\pi\int_0^\pi \rho\sin\theta\sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\,d\theta,$$

since $$ds = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\,d\theta \text{ by Art. 45.}$$

7. Find the surface of the cone obtained by revolving that portion of the line $\frac{x}{a} + \frac{y}{b} = 1$ which is intercepted by the coördinate axes,

(α) about the x-axis; (β) about the y-axis.

8. Find the surface of the sphere obtained by revolving the circle $\rho = 2a\cos\theta$ about the polar axis. [Cf. Ex. 6.]

9. Find the surface generated by the revolution of a complete arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis.

10. Find the surface of the ring generated by revolving the circle $x^2 + (y - k)^2 = a^2$, $k > a$, about the x-axis. Also find the volume of this ring.

11. Find the surface generated by the rotation of the involute of the circle

$$x = a(\cos t + t\sin t), \quad y = a(\sin t - t\cos t)$$

about the x-axis from $t = 0$ to $t = t_1$.

155. Various geometrical problems leading to integration.

Ex. **1.** A string AB of length a has a weight attached at B. The other extremity A moves along a straight line OX, drawing the weight

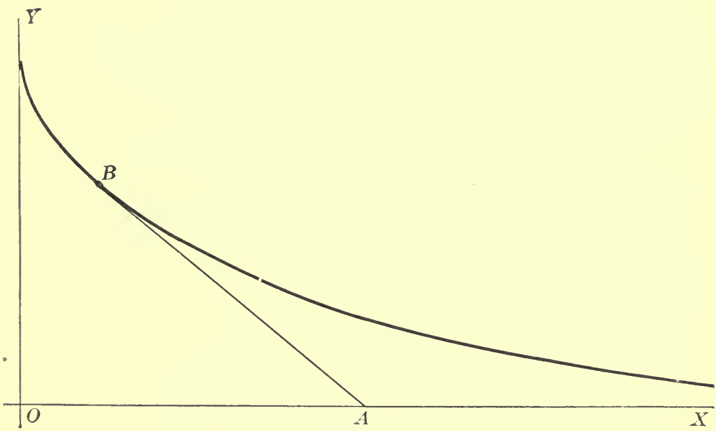

FIG. 80

in a rough horizontal plane XOY. The path traced by the point B is called the *tractrix*. What is its equation?

Let OY be the initial position of the string and AB any intermediate position. Since at every instant the force is exerted on the weight

B in the direction of the string BA, the motion of the point must be in the same direction; that is, the direction of the tractrix at B is the same as that of the line BA and hence BA is tangent to the curve. The expression for the tangent length is (Art. 48, p. 86)

$$\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}} = y\sqrt{\left(\frac{dx}{dy}\right)^2+1} = a.$$

Solving for $\frac{dx}{dy}$, we obtain

$$\frac{dx}{dy} = \sqrt{\frac{a^2-y^2}{y^2}}.$$

Integrating with respect to y gives

$$x = \int \frac{\sqrt{a^2-y^2}}{y}\,dy = \sqrt{a^2-y^2} - a\log\frac{a+\sqrt{a^2-y^2}}{y} + C.$$

The constant of integration is determined by the assumption that $(0, a)$ is the starting point of the curve. Substituting these coördinates in the above equation, we find $C = 0$.

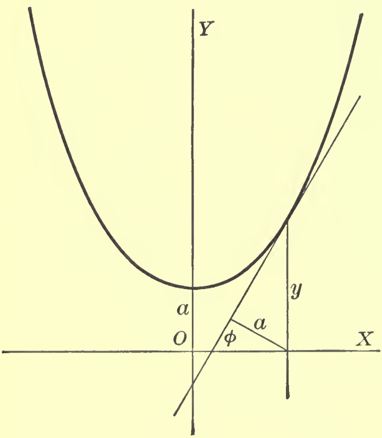

FIG. 81

Ex. **2**. The *equiangular spiral* is a curve so constructed that the angle between the radius vector to any point and the tangent at the same point is constant. Find its equation.

Ex. **3**. Determine the curve having the property that the line drawn from the foot of any ordinate of the curve perpendicular to the corresponding tangent is of constant length a.

If the angle which the

tangent makes with the x-axis is denoted by ϕ, it is at once evident (Fig. 81) that

$$\frac{a}{y} = \cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

From this follows

$$\frac{x}{a} = \log(y + \sqrt{y^2 - a^2}) + C.$$

When the tangent is parallel to the x-axis, the ordinate itself is the perpendicular a. If this ordinate is chosen for the y-axis, the point $(0, a)$ is a point of the curve, and hence

$$C = -\log a.$$

The equation can accordingly be written

$$\frac{y + \sqrt{y^2 - a^2}}{a} = e^{\frac{x}{a}}. \tag{1}$$

From this follows, by taking the reciprocal of both members,

$$\frac{a}{y + \sqrt{y^2 - a^2}} = e^{-\frac{x}{a}},$$

whence, on rationalizing the denominator,

$$\frac{y - \sqrt{y^2 - a^2}}{a} = e^{-\frac{x}{a}}. \tag{2}$$

Adding (1) and (2) and dividing by $\frac{2}{a}$, we obtain

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right),$$

which is the equation of the catenary.

Ex. 4. Find the equation of the curve for which the polar subnormal is proportional to (is a times) the sine of the vectorial angle.

Ex. **5**. Find the equation, in rectangular coördinates, of the curve having the property that the subnormal for any point of the curve is proportional to the abscissa.

Ex. **6**. Find the equation in polar coördinates of the curve for which the angle between the radius vector and the tangent is n times the vectorial angle. What is the curve when $n = 1$? When $n = \frac{1}{2}$?

Ex. **7**. Find the rectangular equation of the curve for which the slope of the tangent varies as the ordinate of the point of contact.

Ex. **8**. Find the equation of the curve for which the polar subtangent is proportional to the length of the radius vector.

Ex. **9**. Find the volume generated by the revolution of the tractrix (see Ex. 1) about the positive x-axis.

Ex. **10**. Find the area of the surface of the revolution described in Ex. 9.

Ex. **11**. Find the length of the tractrix from the cusp (the point $(0, a)$) to the point (x_1, y_1).

Ex. **12**. Derive the following formulas for the length of arc s of a twisted curve, in space of three dimensions, limited by the points (x_1, y_1, z_1), (x_2, y_2, z_2), the coördinates being rectangular:

$$s = \int_{x_1}^{x_2}\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2}\, dx = \int_{y_1}^{y_2}\sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dy$$

$$= \int_{z_1}^{z_2}\sqrt{1 + \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dz = \int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Ex. **13**. Using the formula of Ex. 12, find the length of the helix

$$x = a\cos t, \quad y = a\sin t, \quad z = bt,$$

in which a and b are constants, and t is a variable parameter.

Ex. **14**. A plate of steel is $\frac{1}{4}$ inch thick and has the form of a right segment of a parabola. It weighs 490 lb. per cubic foot. Find the total weight of a plate 30 in. broad and 16 in. long.

Take the equation of the parabola in the form $y^2 = 4px$. Since $y = 15$ when $x = 16$, we may find the value of p by substituting these coördinates in the assumed equation, namely, $4p = \frac{225}{16}$. The area of the parabolic plate is therefore

$$2\int_0^{16} \tfrac{15}{4}\, x^{\frac{1}{2}}\, dx \text{ sq. in.}$$

The volume and hence the weight are now readily obtainable.

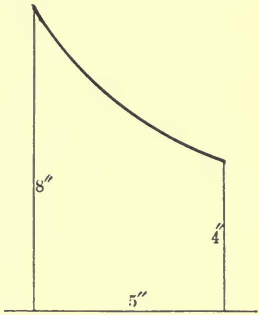

FIG. 82

Ex. **15**. A plate of wrought iron of heaviness 480 lb. per cubic foot is $\frac{1}{8}$ in. thick and is bounded by three straight edges at right angles to each other, as shown in the figure, while the curved boundary is a hyperbola with the equation $(x + 5)\, y = 40$, the base of the figure being on the x-axis. Calculate the weight.

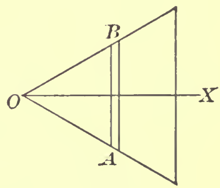

FIG. 83

Ex. **16**. A metal plate, in the form of an equilateral triangle, is $\frac{1}{4}$ in. thick and has an altitude of 4 in. Any very narrow vertical strip, as AB, of length $2y$ and width Δx, is of nearly uniform density. The density varies from one strip to another in such a way that the weight γ per cubic inch is determined by the condition

$$\gamma = 0.26\left(1 + \frac{100}{9 + x^2}\right).$$

Find the weight of the plate.

[HINT. Calculate the weight of the strip AB, then take the limit of the sum of all such strips contained in the figure.]

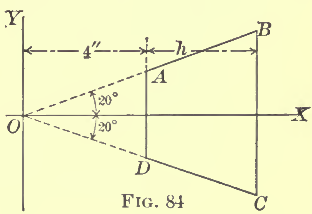

FIG. 84

Ex. **17**. A trapezoidal plate $ABCD$ is $\frac{1}{8}$ in. thick. The weight γ per cubic inch is constant along any vertical line, but varies with x according to the law

$$\gamma = 0.05\, x^{\frac{3}{2}} \text{ oz. per cubic inch.}$$

The first strip DA is 4 in. from the origin. What altitude h must be adopted for the trapezoid in order that the total weight of the plate may be just three ounces?

Ex. **18.** The frustum of a paraboloid of revolution has vertical parallel bases five inches apart. The equation of the meridian curve, with the inch as the linear unit, is $y = \sqrt{x}$. The heaviness γ is constant over a vertical plane section, but varies with x according to the law $\gamma = 0.06\sqrt{100 - x^2}$ lb. per cubic inch. Find the total weight from $x = 4$ to $x = 9$.

CHAPTER VIII

SUCCESSIVE INTEGRATION

156. Functions of a single variable. Thus far we have considered the problem of finding the function y of x when $\frac{dy}{dx}$ only is given. It is now proposed to find y when its nth derivative $\frac{d^ny}{dx^n}$ is given.

The mode of procedure is evident. First find the function $\frac{d^{n-1}y}{dx^{n-1}}$ which has $\frac{d^ny}{dx^n}$ for its derivative. Then, by integrating the result, determine $\frac{d^{n-2}y}{dx^{n-2}}$, and so on until after n successive integrations the required result is found. As an arbitrary constant should be added after each integration in order to obtain the most general solution, the function y will contain n arbitrary constants.

Ex. **1.** Given $\frac{d^3y}{dx^3}=\frac{1}{x^3}$, find y.

Integration of $\frac{1}{x^3}$ with respect to x gives

$$\frac{d^2y}{dx^2}=-\frac{1}{2\,x^2}+C_1.$$

A second integration gives,

$$\frac{dy}{dx}=\frac{1}{2\,x}+C_1x+C_2;$$

and finally $$y=\tfrac{1}{2}\log x+\tfrac{1}{2}C_1x^2+C_2x+C_3.$$

The triple integration required in this example will be symbolized by

$$\int\int\int \frac{1}{x^3}(dx)^3,$$

which will be called the *triple integral* of $\frac{1}{x^3}$ with respect to x.

Ex. 2. Determine the curves having the property that the radius of curvature at any point P is proportional to the cube of the secant of the angle which the tangent at P makes with a fixed line.

If a system of rectangular axes is chosen with the given line for x-axis, it follows from equation (6), p. 173, and from Art. 42, that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{1}{a}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}},$$

in which a is an arbitrary constant. This equation reduces to

$$\frac{d^2y}{dx^2} = a,$$

from which follows

$$y = \int\int a(dx)^2 = a\left[\frac{x^2}{2} + C_1 x + C_2\right],$$

C_1 and C_2 being constants of integration. Hence the required curves are the parabolas having axes parallel to the y-axis.

The existence of the two arbitrary constants C_1, C_2 in the preceding equation makes it possible to impose further conditions. Suppose, for example, it be required to determine the curve having the property already specified, and having besides a maximum (or a minimum) point at (1, 0).

Since at such a point $\frac{dy}{dx} = 0$, it follows that

$$0 = a(1 + C_1),$$

whence $$C_1 = -1.$$

Also, by substituting (1, 0) in the equation of the curve,

$$0 = a(\tfrac{1}{2} - 1 + C_2),$$

from which $$C_2 = \tfrac{1}{2}.$$

Accordingly the required curve is

$$y = \frac{a}{2}(x - 1)^2.$$

Ex. **3.** Find the equation (in rectangular coördinates) of the curves having the property that the radius of curvature is equal to the cube of the tangent length.

[Hint. Take y as the independent variable.]

Ex. **4.** A particle moves along a path in a plane such that the slope of the line tangent at the moving point changes at a rate proportional to the reciprocal of the abscissa of that point. Find the equation of the curve.

Ex. **5.** A particle starting at rest from a point P moves under the action of a force such that the acceleration (cf. Ex. 14, p. 77) at each instant of time is proportional to (is k times) the square root of the time. How far will the particle move in the time t?

Ex. **6.** In connection with a certain curve referred to rectangular axes, we know in advance that it passes through a point A on the y-axis at a distance 1.12 in. above the origin. It also passes through a point B of the first quadrant which is at a distance of 12 in. from the y-axis, and the slope of the tangent to the curve at this point is 0.09. At each point P of the curve the second derivative of y satisfies the relation

$$\frac{d^2y}{dx^2} = 0.0012\,x.$$

It is required to find the general expression (in terms of x) of the ordinate and the slope of the tangent line for any point P of the curve. In particular, find the ordinate and slope when $x = 20$ in.

Ex. **7**. For a certain curve ADN situated in the first quadrant we have given

$$1000\frac{d^2y}{dx^2} = 1.5 - 0.276\,x.$$

The point A has the coördinates (0, 0.04) and the abscissa of D is 10. At the point B of the curve, whose abscissa is 5, the slope of the tangent line is 0.002.

A second curve DC is tangent to the first at the point D, and for each point of it we know that

$$1000\frac{d^2y}{dx^2} = 0.2\,x - 0.115.$$

Find the equations of both curves.

157. Integration of functions of several variables. When functions of two or more variables are under consideration, the process of differentiation can in general be performed with respect to any one of the variables, while the others are treated as constant during the differentiation. A repetition of this process gives rise to the notion of successive partial differentiation with respect to one or several of the variables involved in the given function. [Cf. Arts. 62, 67.]

The reverse process readily suggests itself, and presents the problem: *Given a partial (first, or higher) derivative of a function of several variables with respect to one or more of these variables, to find the original function.*

This problem is solved by means of the ordinary processes of integration, but the added constant of integration has a new meaning. This can be made clear by an example.

Suppose u is an unknown function of x and y such that

$$\frac{\partial u}{\partial x} = 2\,x + 2\,y.$$

Integrate this with respect to x alone, treating y at the same time as though it were constant. This gives

$$u = x^2 + 2xy + \phi,$$

in which ϕ is an added constant of integration. But since y is regarded as constant during this integration, there is nothing to prevent ϕ from depending on it. This dependence may be indicated by writing $\phi(y)$ in the place of ϕ. Hence the most general function having $2x + 2y$ for its partial derivative with respect to x is

$$u = x^2 + 2xy + \phi(y),$$

in which $\phi(y)$ is *an entirely arbitrary function of y.*

Again, suppose

$$\frac{\partial^2 u}{\partial x \partial y} = x^2y^2.$$

Integrating first with respect to y, x being treated as though it were constant during this integration, we find

$$\frac{\partial u}{\partial x} = \frac{1}{3}x^2y^3 + \psi(x),$$

where $\psi(x)$ is an arbitrary function of x, and is to be regarded as an added constant for the integration with respect to y.

Integrate the result with respect to x, treating y as constant. Then

$$u = \tfrac{1}{9}x^3y^3 + \Psi(x) + \Phi(y).$$

Here $\Phi(y)$, the constant of integration with respect to x, is an arbitrary function of y, while

$$\Psi(x) = \int \psi(x)\,dx.$$

Since $\psi(x)$ is an arbitrary function of x, so also is $\Psi(x)$.

158. Integration of a total differential. The total differential of a function u depending on two variables has been defined (Art. 63) by the formula

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

The question now presents itself: *Given a differential expression of the form*

$$P\,dx + Q\,dy, \tag{1}$$

wherein P and Q are functions of x and y, does there exist a function u of the same variables having (1) *for its total differential?*

It is easy to see that in general such a function does not exist. For, in order that (1) may be a total differential of a function u, it is evidently necessary that P and Q have the forms

$$P = \frac{\partial u}{\partial x}, \quad Q = \frac{\partial u}{\partial y}. \tag{2}$$

What relation, then, must exist between P and Q in order that the conditions (2) may be satisfied? This is easily found as follows. Differentiate the first equation of 2 with respect to y, and the second with respect to x. This gives

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y},$$

from which follows (Art. 68)

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{3}$$

This is the relation sought.

The next step is to find the function u by integration. It is easier to make this process clear by an illustration.

Given $\quad (2x+2y+2)\,dx+(2y+2x+2)\,dy,$

find the function u having this as its total differential.

Since $\quad P=2x+2y+2,\ \ Q=2y+2x+2,$

it is found by differentiation that

$$\frac{\partial P}{\partial y}=2 \text{ and } \frac{\partial Q}{\partial x}=2,$$

hence the necessary relation (3) is satisfied.

From (2) it follows that

$$\frac{\partial u}{\partial x}=2x+2y+2.$$

Integrating this with respect to x alone gives

$$u=x^2+2xy+2x+\phi(y). \tag{4}$$

It now remains to determine the function $\phi(y)$ so that

$$\frac{\partial u}{\partial y}[=Q]=2y+2x+2. \tag{5}$$

Differentiating (4) with respect to y alone gives

$$\frac{\partial u}{\partial y}=2x+\phi'(y),$$

where $\phi'(y)$ denotes the derivative of $\phi(y)$ with respect to y. The comparison of this result with (5) gives

$$2y+2x+2=2x+\phi'(y),$$

or

$$\phi'(y)=2y+2, \tag{6}$$

whence, by integrating with respect to y,

$$\phi(y)=y^2+2y+C,$$

in which C is an arbitrary constant with respect to both x and y.

Hence $\quad u=x^2+2xy+2x+y^2+2y+C.$

EXERCISES

Determine in each of the following cases the function u having the given expression for its total differential:

1. $y\,dx + x\,dy$.

2. $\sin x \cos y\,dx + \cos x \sin y\,dy$.

3. $y\,dx - x\,dy$.

4. $\dfrac{y\,dx - x\,dy}{xy}$.

5. $(3x^2 - 3ay)dx + (3y^2 - 3ax)\,dy$.

6. $\dfrac{y\,dx}{x^2 + y^2} - \dfrac{x\,dy}{y^2 + x^2}$.

7. $(2x^2 + 2xy + 5)\,dx + (x^2 + y^2 - y)\,dy$.

8. $(x^4 + y^4 + x^2 - y^2)\,dx + (4y^3x - 2xy + y - y^2 + 2)\,dy$.

159. Multiple integrals. The integration of $\dfrac{\partial^2 u}{\partial x\,\partial y}$ was considered in Art. 157. If $F(x, y)$ is written for the given function, the required integration will be represented by the symbol

$$u = \int\int F(x, y)\,dx\,dy,$$

and the function sought will be called the *double integral* of $F(x, y)$ with respect to x and y.

Likewise
$$\int\int\int F(x, y, z)\,dx\,dy\,dz$$

will be called the *triple integral* of $F(x, y, z)$. It represents the function u whose third partial derivative $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$ is the given function $F(x, y, z)$. It will be understood in what follows that the order of integration is from left to right, that is, we integrate first with respect to the left-hand variable x, then with respect to y, and lastly with respect to z.

Such integrals (double, triple, etc.) will be referred to in general as multiple integrals.

160. Definite multiple integrals. The idea of a multiple integral may be further extended so as to include the notion of a *definite multiple integral* in which limits of integration may be assigned to each variable.

Thus the integral $\int_a^b \int_0^2 x^2y^3\,dy\,dx$ will mean that x^2y^3 is to be integrated first with respect to y between the limits 0 and 2. This gives

$$\int_0^2 x^2y^3\,dy = 4x^2.$$

The result so obtained is to be integrated with respect to x between the limits a and b, which leads to

$$\int_a^b 4x^2\,dx = \tfrac{4}{3}(b^3 - a^3)$$

as the value of the given definite double integral.

In general the expression

$$\int_a^{a'} \int_b^{b'} F(x, y)\,dy\,dx$$

will be used as the symbol of a definite double integral. It will be understood that the integral signs with their attached limits are always to be read from right to left, so that in the above integral the limits for y are b and b', while those for x are a and a'.

Since x is treated as constant in the integration with respect to y, the limits for y may be functions of x. Consider, for example, the integral $\int_0^1 \int_{\sqrt{x}}^{x} xy\,dy\,dx$. The first integration (with respect to y) gives

$$\int_{\sqrt{x}}^{x} xy\,dy = x\left[\frac{y^2}{2}\right]_{\sqrt{x}}^{x} = x\left(\frac{x^2}{2} - \frac{x}{2}\right) = \frac{x^3 - x^2}{2}.$$

By integrating this result with respect to x between limits 0 and 1 the given integral is found to have the value $-\frac{1}{24}$.

EXERCISES

Evaluate the following definite integrals:

1. $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \int_0^2 x \cos(xy)\,dy\,dx.$

2. $\int_0^2 \int_0^x x^2\,dy\,dx.$

3. $\int_1^e \int_0^{\log y} \frac{dx\,dy}{y}.$

4. $\int_1^e \int_0^{\frac{\pi}{4x}} \sec^2(xy)\,dy\,dx.$

5. $\int_0^{\pi} \int_0^{a(1+\cos\theta)} \rho^2 \sin\theta\,d\rho\,d\theta.$

6. $\int_0^b \int_y^{10y} \sqrt{xy - y^2}\,dx\,dy.$

7. $\int_1^2 \int_0^z \int_0^{x\sqrt{3}} \frac{x\,dy\,dx\,dz}{x^2 + y^2}.$

8. $\int_0^1 \int_x^{\sqrt{x}} \int_0^{x+y} \frac{dz\,dy\,dx}{x+y+z}.$

161. Plane areas by double integration. The area bounded by a plane curve (or by several curves) can be readily expressed in the form of a definite double integral. An illustrative example will explain the method.

Ex. 1. Find by double integration the area of the circle

$$(x-a)^2 + (y-b)^2 = r^2.$$

Imagine the given area divided into rectangles by a series of lines parallel to the y-axis at equal distances Δx, and a series of lines parallel to the x-axis at equal distances Δy.

The area of one of these rectangles is $\Delta y \cdot \Delta x$. This is called the element of area. The sum of all the rectangles interior to the circle will be less than the area required by the amount contained in the small subdivisions which border the circumference of the circle.

All these neglected portions are contained within a ring bounded by the given circle and a circle concentric with it, whose radius is less than r by the length of diagonal of an element of area, that is, of radius $r - \sqrt{(\Delta x)^2 + (\Delta y)^2}$. In other words, the amount neglected is less than the area of a circular ring whose width is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

and which therefore approaches zero simultaneously with Δx and Δy. Hence the area of the circle is the limit of the sum of all the elements of area included within it.

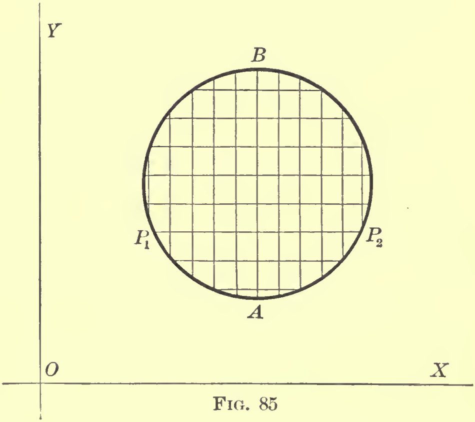

Fig. 85

To find the value of the limit of this sum it is convenient first to add together all the elements contained between two consecutive parallels. Let P_1P_2 be one of these parallels having the direction of the x-axis. Then y remains constant while x varies from $a - \sqrt{r^2 - (y-b)^2}$ (the value of the abscissa at P_1) to $a + \sqrt{r^2 - (y-b)^2}$ (the value at P_2). The limit, as Δx approaches zero, of the sum of rectangles in the strip from P_1P_2 is evidently

$$\Delta y[\text{limit of sum } (\Delta x + \Delta x + \cdots)] = \Delta y \int_{a-\sqrt{r^2-(y-b)^2}}^{a+\sqrt{r^2-(y-b)^2}} dx. \qquad (1)$$

Now find the limit of the sum of all such strips contained within the circle. This requires the determination of the limit of the sum of terms such as (1) for the different values of y corresponding to the different strips. Since y begins at the lowest point A with the value $b-r$, and increases to $b+r$, the value reached at B, the final expression for the area is

$$\int_{b-r}^{b+r}\left[\int_{a-\sqrt{r^2-(y-b)^2}}^{a+\sqrt{r^2-(y-b)^2}} dx\right]dy = \int_{b-r}^{b+r}\int_{a-\sqrt{r^2-(y-b)^2}}^{a+\sqrt{r^2-(y-b)^2}} dx\,dy.$$

Integrating first with respect to x gives

$$\int_{a-\sqrt{r^2-(y-b)^2}}^{a+\sqrt{r^2-(y-b)^2}} dx = x\Big]_{a-\sqrt{r^2-(y-b)^2}}^{a+\sqrt{r^2-(y-b)^2}} = 2\sqrt{r^2-(y-b)^2}.$$

This result is then integrated with respect to y, giving

$$\int_{b-r}^{b+r} 2\sqrt{r^2-(y-b)^2}\,dy = (y-b)\sqrt{r^2-(y-b)^2} + r^2\sin^{-1}\frac{y-b}{r}\Big]_{b-r}^{b+r} = \pi r^2.$$

If the summation had begun by adding the rectangles in a strip parallel to the y-axis, and then adding all of these strips, the expression for the area would take the form

$$\int_{a-r}^{a+r}\int_{b-\sqrt{r^2-(x-a)^2}}^{b+\sqrt{r^2-(x-a)^2}} dy\,dx.$$

It is seen from the last result that the order of integration in a double integral can be changed if the limits of integration are properly modified at the same time.

Ex. **2.** Find the area which is included between the two parabolas

$$y^2 = 9x \text{ and } y^2 = 72 - 9x.$$

Ex. **3.** Find the area between $y^2 = 5x$ and $y = x$.

Ex. **4.** Find by double integration the area of the segment of the circle $x^2 + y^2 = 16$ cut off by the line $x + y = 4$.

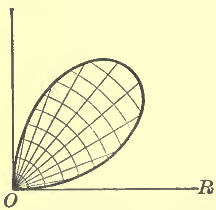

Fig. 86

Ex. **5.** Find the area between the two curves $y^3 = x$ and $y = x^3$.

Ex. **6.** Find the area between the two curves $y^2 = x^3$ and $y^2 = x$.

Ex. **7.** Find by double integration the area of one loop of the polar curve $\rho = a\sin 2\theta$.

Imagine the area divided into small elements by means of concentric circles whose radii vary by equal increments $\Delta\rho$ and by means of radii drawn from the origin, the angle between two consecutive radii being $\Delta\theta$. (See Fig. 86.)

The area of an arbitrary element may be expressed as the difference of two circular sectors with a common angle $\Delta\theta$ and with radii $\rho + \Delta\rho$ and ρ respectively. That is,

$$\begin{aligned}\text{element of area} &= \tfrac{1}{2}(\rho + \Delta\rho)^2 \Delta\theta - \tfrac{1}{2}\rho^2 \Delta\theta \\ &= \rho\Delta\theta\Delta\rho + \tfrac{1}{2}\Delta\theta\,(\Delta\rho)^2.\end{aligned}$$

The sum of all the complete elements within the loop may then be represented by the formula

$$\sum \rho\,\Delta\theta\,\Delta\rho + \tfrac{1}{2}\sum \Delta\theta\,(\Delta\rho)^2.$$

Reasoning precisely as in Ex. 1, we find the limit of the first sum to be

$$\int_0^{\frac{\pi}{2}}\int_0^{a \sin 2\theta} \rho\, d\rho\, d\theta.$$

The second sum may be written $\frac{1}{2}\Delta\rho\Sigma\,\Delta\theta\,\Delta\rho$, hence its limit is

$$\tfrac{1}{2}\cdot \lim \Delta\rho \cdot \lim \sum \Delta\theta\,\Delta\rho = \tfrac{1}{2}\cdot 0 \cdot \int_0^{\frac{\pi}{2}}\int_0^{a \sin 2\theta} d\rho\, d\theta = 0.$$

Following the analogy of Ex. 1, we can easily see that all the neglected incomplete elements of area lie within a narrow band along the boundary of the given area, the width of which band approaches 0. Their sum therefore approaches zero in passing to the limit.

It follows from the preceding discussion that the general formula for area in polar coördinates is

$$\int\int \rho\, d\rho\, d\theta,$$

the limits of integration being determined by the boundary of the given area.

Ex. **8.** Find by double integration the area of the cardioid

$$\rho = a(1 - \cos\theta).$$

Ex. **9**. Find the area of the lemniscate $\rho^2 = a^2 \cos 2\theta$.

Ex. **10**. Express by double integrals the three areas between the cardioid (Ex. 8) and the circle $\rho = a$.

Ex. **11**. Find by double integration the area of the triangle whose vertices have the rectangular cöordinates (5, 2), (– 3, 6), (7, 6).

Ex. **12**. Find the area common to the two circles

$$x^2 - 8x + y^2 - 8y + 28 = 0,$$

$$x^2 - 8x + y^2 - 4y + 16 = 0.$$

162. Volumes. The volume bounded by one or more surfaces can be expressed as a triple integral when the equations of the bounding surfaces are given.

Let it be required to find the volume bounded by the surface ABC (Fig. 87) whose equation is $z = f(x, y)$, and by the three coördinate planes.

Imagine the figure divided into small equal rectangular parallelopipeds by means of three series of planes, the first series parallel to the yz-plane at equal distances Δx, the second parallel to the xz-plane at equal distances Δy, and the third parallel to the xy-plane at equal distances Δz. The volume of such a rectangular solid is $\Delta x\, \Delta y\, \Delta z$; it is called the element of volume. The limit of the sum of all such elements contained in $OABC$ is the volume required, provided that the bounding surface ABC is continuous. For the sum of the neglected incomplete elements, which border the surface, is less than the volume of a shell whose outside boundary is the given surface and whose thickness is $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$, the diagonal of the element of volume. Hence the error approaches zero as the three increments diminish.

To effect this summation, add first all the elements in a vertical column. This corresponds to integrating with respect to z (x and y remaining constant) from zero to $f(x, y)$. Then add all such vertical columns contained between two consecutive planes parallel to the yz-plane (x remaining constant), which corresponds to an integration with respect to y from $y=0$ to the value attained on the boundary of the curve AB.

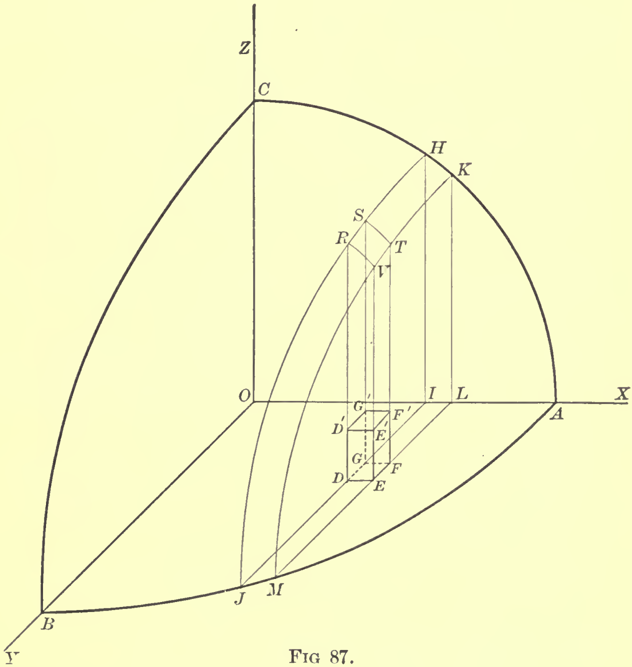

FIG 87.

This value of y is found by solving the equation $f(x, y)=0$. Finally, add all such plates for values of x varying from zero to its value at A. The result is expressed by the integral

$$\int_0^a \int_0^{\phi(x)} \int_0^{f(x,y)} dz\, dy\, dx,$$

in which $\phi(x)$ is the result of solving the equation $f(x, y) = 0$ for y, and a is the x-coördinate of A.

Ex. **1**. Find the volume of the sphere of radius a.

The equation of the sphere is

$$x^2 + y^2 + z^2 = a^2,$$

or

$$z = \sqrt{a^2 - x^2 - y^2}.$$

Since the coördinate planes divide the volume into eight equal portions, it is sufficient to find the volume in the first octant and multiply the result by 8.

The volume being divided into equal rectangular solids as described above, the integration with respect to z is equivalent to finding the limit of the sum of all the elements contained in any vertical column. The limits of the integration with respect to z are the values of z corresponding to the bottom and the top of such a column, namely, $z = 0$, and $z = \sqrt{a^2 - x^2 - y^2}$, since the point at the top is on the surface of the sphere.

The limits of integration with respect to y are found to be $y = 0$ (the value at the x-axis), and $y = \sqrt{a^2 - x^2}$ (the value of y at the circumference of the circle $a^2 - x^2 - y^2 = 0$, in which the sphere is cut by the xy-plane).

Finally, the limiting values for x are zero and a, the latter being the distance from the origin to the point in which the sphere intersects the x-axis. Hence

$$V[= \text{volume of sphere}] = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx.$$

Integration with respect to z gives

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx;$$

then with respect to y and x,

$$V = 8\int_0^a dx \left[\frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{a^2 - x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2 - x^2}}\right]_0^{\sqrt{a^2-x^2}}$$

$$= 8\int_0^a \frac{\pi}{4}(a^2 - x^2)dx = \frac{4\pi a^3}{3}.$$

Ex. **2**. Find the volume of one of the wedges cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = mx$.

Ex. **3**. Find the volume common to two right circular cylinders of the same radius a whose axes intersect at right angles.

Ex. **4**. Find the volume of the cylinder $(x - 1)^2 + (y - 1)^2 = 1$ limited by the plane $z = 0$, and the hyperbolic paraboloid $z = xy$.

Ex. **5**. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Ex. **6**. Find the volume of that portion of the elliptic paraboloid

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

which is cut off by the plane $z = 0$.

Ex. **7**. Find by triple integration the volume of the tetrahedron formed by the three coördinate planes and the plane $x + 2y + 3z = 1$.

Ex. **8**. Find the volume of the elliptic paraboloid $2y^2 + 3z^2 = 6x$ cut off by the plane $x = 2$.

CHAPTER IX

SOME APPLICATIONS OF INTEGRAL CALCULUS TO PROBLEMS OF MECHANICS

163. Liquid pressure on a plane vertical wall. The pressure exerted by the liquid upon any point of a plane vertical wall is proportional to the depth of that point below the surface of the fluid. To calculate the pressure upon the entire wall we divide it into narrow horizontal strips of equal areas ΔA. Denote the breadth of the kth strip PQ (Fig. 88), counting from the top, by h_k. The pressure exerted on the kth strip is equivalent to the weight of a column of fluid standing on a base of the same area ΔA and having an altitude intermediate between the least depth x and the greatest depth $x + h_k$ of points on the given strip. This altitude may be represented by $x + \theta_k h_k$ in which θ_k has a value between 0 and 1. If w denotes the weight of a cubic unit of the fluid, the pressure on PQ is $w(x + \theta_k h_k)\ \Delta A$. Summing the pressures for all the strips of the wall, we obtain for the total pressure

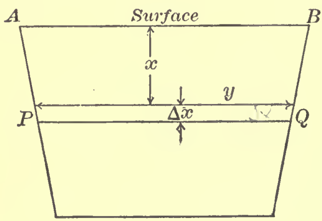

FIG. 88.

$$\sum w(x + \theta_k h_k)\,\Delta A.$$

In order to evaluate this sum we take its limit as ΔA approaches zero. This gives, by separating into two terms and observing that w is constant,

$$w \lim_{\Delta A \doteq 0} \sum x \, \Delta A + w \lim_{\Delta A \doteq 0} \sum \theta_k h_k \, \Delta A.$$

The second term reduces to zero. For,

$$\sum \theta_k h_k \, \Delta A = \Delta A \sum \theta_k h_k < \Delta A \cdot H \text{ (since } \theta_k < 1\text{)},$$

in which H denotes the total altitude of the wall; as $\Delta A \doteq 0$ the right member of this inequality approaches zero. Hence

$$\text{pressure} = w \int x \, dA.$$

In order to evaluate the integral, it is most convenient to make x the variable of integration. Denote by y the width of the wall at the depth x. Then $\Delta A = y_k \, \Delta x$ in which y_k is a certain value of y between y and $y + \Delta y$. (Compare Art. 40.) Dividing by Δt and passing to the limit we obtain, since $\lim y_k = y$,

$$\frac{dA}{dt} = y \, \frac{dx}{dt},$$

or in the differential notation, $dA = y \, dx$. The substitution of this in the above integral gives

$$\text{pressure} = w \int xy \, dx,$$

the limits of integration being the values of x at the top and the bottom of the given wall or surface.

If the liquid is water and the unit of length is a foot, then $w = 62\frac{1}{4}$ lb.

EXERCISES

1. Find the pressure on the end of a rectangular tank full of water that is 10 ft. long, 8 ft. wide, and 5 ft. deep.

2. A watermain 6 ft. in diameter is half full of water. Find the pressure on the gate that closes the main.

3. A vertical masonry dam in the form of a trapezoid is 200 ft. long at the surface of the water, 150 ft. long at the bottom, and 60 ft. high. What pressure must it withstand?

4. A vertical cross section of a trough is a parabola with vertex downwards, the latus rectum lying in the surface and being 4 ft. long. Find the pressure on the end of the trough when it is full of water.

5. One end of an unfinished watermain, 4 ft. in diameter, is closed by a temporary bulkhead and the water is let in from the reservoir. Find the pressure on the bulkhead if its center is 40 ft. below the surface of the water in the reservoir.

164. Center of gravity. (1) *For a system of n particles.* Let P_1, P_2 be two particles of matter of masses (or weights) m_1 and m_2, respectively, and let x_1, x_2 be their distances from a chosen point O on the straight line through them.

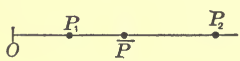

FIG. 89

There exists a point $\bar{P}$ such that the segments $P_1\bar{P}$ and $\bar{P}P_2$ are inversely proportional to the masses of the two points, that is,

$$\frac{P_1\bar{P}}{\bar{P}P_2}=\frac{m_2}{m_1}. \tag{1}$$

Let $\bar{x}$ represent the distance $O\bar{P}$. Then formula (1), expressed in terms of the abscissas of the points, is

$$\frac{\bar{x}-x_1}{x_2-\bar{x}}=\frac{m_2}{m_1},$$

whence, by solving for $\bar{x}$,

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \tag{2}$$

The point $\bar{P}$ is called *the center of gravity*, or, *the center of mass*, of the system formed by the two points P_1, P_2. If we imagine the line P_1P_2 to consist of a rigid, weightless rod with the two given particles fastened at its extremities, and if we suppose this object to rest on the point $\bar{P}$ as a base, it will remain in equilibrium, without any tendency in either of the end points to move downward under the force of gravity.

In other words, the system of two particles is equivalent, as far as the action of gravity is concerned, to a single particle, of mass $m_1 + m_2$, placed at the point $\bar{P}$.

Let P_3 be a third point of mass m_3 situated on the same line with P_1 and P_2. Then the abscissa $\bar{x}$ of the center of gravity of the system of three points may be found by calculating the center of gravity of the pair P_3 and $\bar{P}$ (the center of gravity for P_1, P_2), the mass of $\bar{P}$ being taken as $m_1 + m_2$, the sum of the masses of P_1 and P_2. This gives

$$\bar{x} = \frac{(m_1 + m_2)\dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2} + m_3 x_3}{(m_1 + m_2) + m_3} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}.$$

In like manner the center of gravity for any number n of particles situated on a straight line is given by the formula

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}. \tag{3}$$

If the n particles are not on a straight line but are situated in the same plane at the points (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n), then the center of gravity of the system has its abscissa given by (3) and its ordinate $\bar{y}$ is

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n}.$$

If the n particles are not situated in one plane, there will be a third and similar formula for $\bar{z}$.

(2) *For a continuous solid.* Imagine the solid divided up into small elements, precisely as in determining its volume, by means of three series of planes parallel to the coördinate planes and at distances Δx, Δy, Δz. If we regard any particular element as being very nearly of uniform density, then the mass of an arbitrary element is approximately $\rho\, \Delta x\, \Delta y\, \Delta z$, in which ρ is the weight of a cubic unit of homogeneous matter having the same density as the given element. This number ρ is usually called the density. For a finite number of elements the x-coördinate of the center of gravity is determined approximately by (3) in the form

$$\frac{(\rho_1 x_1 + \rho_2 x_2 + \cdots + \rho_n x_n)\, \Delta x\, \Delta y\, \Delta z}{(\rho_1 + \rho_2 + \cdots + \rho_n)\, \Delta x\, \Delta y\, \Delta z},$$

in which $x_1, x_2, \cdots$ are the abscissas for the different elements and $\rho_1, \rho_2, \cdots$ are their densities. The abscissa of the center of gravity of the given continuous solid is obtained by making Δx, Δy, Δz approach zero as a limit.* This gives

$$\bar{x} = \frac{\int\int\int \rho\, x\, dx\, dy\, dz}{\int\int\int \rho\, dx\, dy\, dz},$$

* A proof of this statement will be found in Art. 166.

the limits of integration being determined just as in calculating the volume of the solid. If the solid is homogeneous, ρ is constant and cancels out of numerator and denominator. Otherwise, it is a function of x, y, z.

In precisely the same manner the values of $\bar{y}$ and $\bar{z}$ are obtained. The coördinates of the center of gravity are thus found to be

$$\bar{x} = \frac{1}{M}\iiint \rho\, x\, dx\, dy\, dz, \quad \bar{y} = \frac{1}{M}\iiint \rho\, y\, dx\, dy\, dz,$$

$$\bar{z} = \frac{1}{M}\iiint \rho\, z\, dx\, dy\, dz,$$

in which ρ is the density at the point (x, y, z) and M is the total mass of the given solid, that is,

$$M = \iiint \rho\, dx\, dy\, dz.$$

The coördinates of the center of gravity of a plane area are found in like manner to be

$$\bar{x} = \frac{1}{M}\iint \rho\, x\, dx\, dy, \quad \bar{y} = \frac{1}{M}\iint \rho\, y\, dx\, dy, \quad M = \iint \rho\, dx\, dy.$$

EXERCISES

In the following problems ρ is understood to be constant unless otherwise specified. The abbreviation C. G. will be used for "center of gravity."

1. Find the C. G. of the tetrahedron whose faces are the three cöordinate planes and the plane $x + 2y + 3z = 6$.

2. Find the C. G. of the volume bounded by the coördinate planes the plane $x + y = 1$, and the surface $z = xy$.

3. Find the C. G. of the volume bounded by the hyperboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ and the plane $x = k$, $k > a$.

4. Find the C. G. of the semiellipsoid on the positive side of the xy-plane, the equation of the ellipsoidal surface being $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

5. Find the C. G. of a thin hemispherical shell of thickness h bounded by two concentric hemispheres of radii a and $a + h$.

6. A hemispherical iron bowl of uniform thickness a is filled with water. If the density of iron is seven times that of water, find the C. G., supposing the radius of the interior of the bowl to be r.

[HINT. Find the C. G. of the iron bowl by means of Ex. 5. Find the C. G. of the hemisphere of water and combine the centers of gravity of the iron and the water by means of (2).]

7. Show that the C. G. of a triangular plate one inch thick is one half inch below the intersection of the medians of the upper face.

8. Find the C. G. of a T-iron one inch thick, the vertical bar being a inches wide and b inches high, and the horizontal bar a' inches wide and b' inches long.

9. Find the C. G. of a sector of a circle of radius a and angle θ.

10. Find the C. G. of the segment of the circle $x^2 + y^2 = r^2$ cut off by the line $x = a$, $0 < a < r$.

11. Find the C. G. of the quadrant of an ellipse.

12. Find the C. G. of the segment of an ellipse cut off by the chord joining the extremities of the major and minor axes.

13. Find the C. G. of the area bounded by the parabola

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

and the line

$$x + y = a.$$

14. Prove that the volume of a solid of revolution is equal to the product of the generating area by the length of path described by its center of gravity.

15. Find the C. G. of an octant of an ellipsoidal mass.

16. Find the C. G. of the preceding mass when the density varies directly as the distance from the plane $x = 0$.

17. Find the C. G. of an octant of a sphere. From this result find the C. G of an octant of a spherical shell of thickness h and inner radius a.

18. Find the C. G. of an octant of a sphere if the density varies directly as the distance from the center of the sphere.

[HINT. Divide up into thin concentric shells of equal thickness h, the density of a particular shell being regarded as constant. Let λ denote the radius of an arbitrary shell, $\bar{\lambda}$ the distance of its C. G. from the origin, and m its mass. Calculate $\bar{\lambda}$ in terms of λ by means of Ex. 17, measuring it on a line equally inclined to the x, y, z axes. Then use the different values of $\bar{\lambda}$ in place of x_1, x_2, $\cdots$, formula (3), and pass to the limit.

19. Find the C. G. of a right circular cone of altitude h and base-radius r.

This problem can be solved by single integration if we suppose the solid divided up into thin plates of equal thickness by means of planes parallel to the base. Then find the approximate expression for the C. G. of any plate, apply (3), and pass to the limit.

20. Find the C. G. of the portion of the elliptical cone $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = (z-1)^2$ between the vertex (0, 0, 1) and the xy-plane.

21. A cone of vertical angle $2\,\theta$ has its vertex on the surface of a sphere, its axis passing through the center of the sphere.

(a) Find the C. G. of the mass outside the cone and inside the sphere.

(b) Find the C. G. of the mass inside the sphere and inside the cone.

165. Moment of Inertia. The moment of inertia of a small particle of matter of mass m about an axis is defined as the product of the mass by the square of the distance of the particle from the axis. It measures the resistance of the particle to rotation about the axis.

To find the moment of inertia of a homogeneous solid body, imagine it divided up into small rectangular blocks (or elements) of dimensions Δx, Δy, Δz. Then the moment of inertia of a single element about the x-axis is approximately

$$\rho(y^2 + z^2)\Delta x\, \Delta y\, \Delta z,$$

in which ρ is the density, that is, it is the weight of a cubic unit of the given solid. Summing up these elements over the whole body and taking the limit of the sum, we find the moment of inertia to be *

$$\iiint \rho\,(y^2 + z^2)\,dx\,dy\,dz, \tag{4}$$

the triple integral being extended over the entire solid, just as was done in finding its volume.

If the solid is not homogeneous, then ρ is variable. Its value at a specified point P of the given body is equal to the weight of a homogeneous cubic unit of matter having the same density throughout as the particle of matter at the point P. It is a function of x, y, z which is to be determined by the conditions of the given problem.

Similarly, the moment of inertia of a plane area about the x-axis is defined as the limit of the sum of terms formed by multiplying each element of area by the square of its distance from the axis. This gives the formula

$$\iint y^2\,dx\,dy.$$

* See the next article for a completion of the proof.

EXERCISES

In the following problems M. I. is used for brevity to denote "moment of inertia." Unless the contrary is stated, the body is homogeneous and of density ρ.

1. Find the M. I. of a rectangular parallelopiped of dimensions a, b, c about an edge a.

Take three edges a, b, c meeting in a common point as the x, y, z axes, respectively. Then by formula (4) the M. I. is

$$\rho\int_0^a\int_0^b\int_0^c(y^2+z^2)\,dz\,dy\,dx.$$

2. Find the M. I. of a circular cylinder of radius a and altitude h about its axis.

3. Find the M. I. of the cylinder of Ex. 2 about a line perpendicular to, and bisecting, the axis.

4. Find the M. I. of a circular cone of altitude a and base-radius r about its axis.

[HINT. If the axis of the cone is taken for the x-axis and its vertex at the origin, the equation of the conical surface is

$$\frac{x^2}{a^2}=\frac{y^2+z^2}{r^2}.\ ]$$

5. Find the M. I. of an elliptical right cylinder about its longitudinal axis, the axes of the elliptical bases being $2a$, $2b$ and the altitude h.

6. Find the M. I. of the preceding solid about the minor axis of an elliptical base.

7. Find the M. I. of the same body about a line bisecting the longitudinal axis and parallel to the major axes of the elliptical bases.

8. Find the M. I. of a sphere about a diameter. Hence find the M. I. of a spherical shell of uniform thickness h about a diameter, assuming that the M. I. of a solid consisting of two parts is the sum of the moments of the separate parts.

9. Find the M. I. of a spherical solid of radius r about a diameter if the density varies directly as the nth power of the distance from the center.

[HINT. Imagine the sphere divided into concentric shells of equal thickness $\Delta\lambda$ and denote by λ the interior radius of any shell. Using the preceding problem, write down the element of M. I., that is, the M. I. of the shell of radius λ and thickness $\Delta\lambda$. Take the limit of the sum of all such elements as $\Delta\lambda \doteq 0$. The required M. I. is thus obtained by a single integration.]

10. Find the M. I. of a cube of edge a about its diagonal.

[HINT. Take three faces of the cube as coördinate planes. Obtain an expression for the square of the distance from any point (x, y, z) to the diagonal of the cube that passes through the origin. This, multiplied by $\Delta x \Delta y \Delta z$, will be the element of M. I. Then take the limit of the sum.]

11. Find the M. I. of a cylindrical shell, of length a, about its axis, the radius of the inner surface being r and that of the outer surface being R.

12. Find the M. I. of a rectangle of sides a, b about the side b.

13. Find the M. I. of a triangle of base b and altitude h about an axis through a vertex parallel to the opposite side.

14. Find the M. I. of a circle of radius a about a diameter.

166. Duhamel's Theorem. In order to complete the proof of the formulas for center of gravity and moment of inertia, we make use of the following theorem which is of very general use in applications of the Integral Calculus.

DUHAMEL'S THEOREM. *Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be positive variables, each of which approaches zero as n increases without limit, and suppose that the sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ approaches a finite limit as $n \doteq \infty$. Let $\beta_1, \beta_2, \cdots, \beta_n$ be variables having the same property as the α's and such that $\lim\limits_{n \doteq \infty} \dfrac{\beta_k}{\alpha_k} = 1$ for $k = 1, 2, \cdots, n$. Then*

$$\lim_{n \doteq \infty} (\beta_1 + \beta_2 + \cdots + \beta_n) = \lim_{n \doteq \infty} (\alpha_1 + \alpha_2 + \cdots + \alpha_n).$$

Since $\lim\limits_{n \doteq \infty} \dfrac{\beta_k}{\alpha_k} = 1$, we may write $\dfrac{\beta_k}{\alpha_k}$ in the form $1 + \epsilon_k$ in which ϵ_k approaches zero as $n \doteq \infty$. Hence,

$$\beta_k = \alpha_k + \epsilon_k \alpha_k,$$

and therefore
$$\sum \beta_k = \sum \alpha_k + \sum \epsilon_k \alpha_k.$$

Let ϵ denote the positive value of the numerically greatest term of the series $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$. Then we have the inequalities

$$-\epsilon\alpha_1 \leqq \epsilon_1\alpha_1 \leqq +\epsilon\alpha_1,$$

$$-\epsilon\alpha_2 \leqq \epsilon_2\alpha_2 \leqq +\epsilon\alpha_2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$-\epsilon\alpha_n \leqq \epsilon_n\alpha_n < +\epsilon\alpha_n,$$

and by adding we obtain

$$-\epsilon(\alpha_1 + \alpha_2 + \cdots + \alpha_n) \leqq \sum \epsilon_k\alpha_k \leqq +\epsilon(\alpha_1 + \alpha_2 + \cdots + \alpha_n).$$

Now let n increase without limit. Since by hypothesis $\epsilon \doteq 0$ and $(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$ has a finite limit, it follows that the first and last members of the preceding inequalities vanish at the limit and therefore

$$\lim_{n \doteq \infty} \sum \epsilon_k \alpha_k = 0.$$

Hence

$$\lim_{n \doteq \infty} \sum \beta_k = \lim_{n \doteq \infty} \sum \alpha_k.*$$

As an application of the above theorem, consider the sum occurring in the approximate formula for center of gravity, namely,

$$(\rho_1 x_1 + \rho_2 x_2 + \cdots + \rho_n x_n)\Delta V$$

in which

$$\Delta V = \Delta x \, \Delta y \, \Delta z.$$

Let ρ_k', x_k' be the minimum, and ρ_k'', x_k'' the maximum values of ρ, x in the kth element of volume. For brevity write

$$\rho_k' x_k' \Delta V = \alpha_k, \quad \rho_k'' x_k'' \Delta V = \beta_k.$$

Then we have

$$\alpha_k \leqq \rho_k x_k \Delta V \leqq \beta_k$$

hence, by taking the sum,

$$\sum \alpha_k \leqq \sum \rho_k x_k \Delta V \leqq \sum \beta_k.$$

But $\dfrac{\beta_k}{\alpha_k} = \dfrac{\rho_k'' x_k''}{\rho_k' x_k'}$ which approaches 1 as n increases since ρ_k', x_k' approach equality with ρ_k'', x_k''. Hence

$$\lim \sum \alpha_k = \lim \sum \beta_k = \lim \sum \rho_k x_k \Delta V.$$

In obtaining this result no restriction is placed on x_k and ρ_k

* A variable which has zero as a limit is often called an *infinitesimal*. Hence $\alpha_1, \alpha_2, \cdots, \alpha_n$ are infinitesimals. If we write $\delta_k = \epsilon_k \alpha_k$, then $\lim_{n \doteq \infty} \dfrac{\delta_k}{\alpha_k} = \lim \epsilon_k = 0$. When two infinitesimals, δ and α, are so related that the ratio of δ to α has the limit zero, then δ is said to be infinitesimal with respect to α, or it is called an infinitesimal of a higher order than α.

Since, by Duhamel's Theorem, $\lim \sum(\alpha_k + \delta_k) = \lim \sum \alpha_k$, this theorem is equivalent to saying that *the limit of a sum of infinitesimals is not affected by dropping from each term an infinitesimal of a higher order.*

ANSWERS

DIFFERENTIAL CALCULUS

Page 23. Art. 7

1. $2x-2$; 2; 0; 1.
2. $6x-4$.
3. $-\dfrac{1}{4x^2}$.
4. $4x^3-\dfrac{6}{x^3}$.
5. $-\dfrac{3}{x^4}$.
6. nx^{n-1}.
7. $\dfrac{x^2+2x}{(x+1)^2}$.
8. $\dfrac{1-x^2}{(x^2+1)^2}$.
9. $\dfrac{1}{2\sqrt{x}}$.
10. $-\frac{2}{3}x^{-\frac{5}{3}}$.

Page 24. Art. 8

2. $(6u-4)\,6x^2$.
3. $-\dfrac{1}{u^2}(10x-2)$.
4. $\left(6u-\dfrac{2}{3u^3}\right)\left(x^2-\dfrac{9}{x^4}\right)$.

Page 32. Art. 13

1. $10x^9$.
2. $-8x^{-9}$.
3. $\dfrac{c}{2\sqrt{x}}$.
4. $-\dfrac{1}{\sqrt{x^3}}-\dfrac{1}{9\sqrt[3]{x^2}}$.
5. $-\frac{5}{4}\sqrt[4]{x^{-9}}$.
6. $n(x+a)^{n-1}$.
7. nx^{n-1}.
8. $\dfrac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$.
9. $\dfrac{2-6x-x^2}{(x^2+2)^2}$.
10. $\dfrac{3x+5}{\sqrt{x+2}}$.
11. $\dfrac{\sqrt{a}(\sqrt{x}-\sqrt{a})}{2\sqrt{x}(\sqrt{x+a})(\sqrt{a}+\sqrt{x})^2}$.
12. $\dfrac{1}{\sqrt{1-x^2}(1-x)}$.
13. $\dfrac{1}{2x(1-x^2)+\sqrt{1-x^2}}$.

14. $\dfrac{4a^{\frac{1}{2}}+3x^{\frac{1}{2}}}{4\sqrt{x}\sqrt{a^{\frac{1}{2}}+x^{\frac{1}{2}}}}$.

15. $\dfrac{ny}{x\sqrt{1-y^2}}$.

16. $\dfrac{2x^3-4x}{(1-x^2)^{\frac{1}{2}}(1+x^2)^{\frac{5}{2}}}$.

17. $\dfrac{-2nx^{n-1}}{(x^n-1)^2}$.

18. $-\dfrac{m(b+x)+n(a+x)}{(a+x)^{m+1}\cdot(b+x)^{n+1}}$.

19. $\dfrac{-2}{x^2(x^3+1)^{\frac{5}{3}}}$.

20. $56x^3(x^2+1)^{\frac{1}{3}}$.

21. $6u\dfrac{du}{dx}$.

22. $12(u^2-u+1)\dfrac{du}{dx}$.

23. $60u^5(1+u^2)^2\dfrac{du}{dx}$.

24. $u+x\dfrac{du}{dx}$.

25. $(2u+6xu)\dfrac{du}{dx}+3u^2+4x^3$.

26. $\dfrac{nu^{n-1}\dfrac{du}{dx}}{(a+x)^n}-\dfrac{nu^n}{(a+x)^{n+1}}$.

27. $2ux^3w\dfrac{du}{dx}+u^2x^3\dfrac{dw}{dx}+3u^2x^2w$.

30. $\dfrac{-b^2x}{a^2y}=-\dfrac{bx}{a\sqrt{a^2-x^2}}$.

32. $(0, 0)$, $\left(\dfrac{4}{9a}, -\dfrac{8}{27a}\right)$, $\left(\dfrac{4}{9a}, +\dfrac{8}{27a}\right)$.

34. $\dfrac{(21u^3-19u)10x}{(7u^2+5)^{\frac{3}{2}}}$.

35. At right angles at $(3, \pm 6)$.

Page 33. Art. 14

2. $-\dfrac{y^2+2xy}{2xy+x^2}$.

3. $\dfrac{6(2x-3y)^2+2(x-y)+1}{9(2x-3y)^2+2(x-y)-1}$.

Page 37. Art. 18

1. $\dfrac{1}{x+a}$.

2. $\dfrac{a}{ax+b}$.

3. $\dfrac{8x-7}{4x^2-7x+2}$.

4. $\dfrac{2}{1-x^2}$.

5. $\dfrac{4x}{1-x^4}$.

6. $\log x+1$

7. $nx^{n-1}\log x+x^{n-1}$.

8. $nx^{n-1}\log x^m+mx^{n-1}$.

9. $\dfrac{x}{x^2-1}$.

10. $\dfrac{1}{2(\sqrt{x}+1)}$.

11. $\log_a e\cdot\dfrac{12x\sqrt{2+x}-1}{2\sqrt{2+x}(3x^2-\sqrt{2+x})}$.

12. $\log_{10} e\cdot\dfrac{2x+7}{x^2+7x}$.

13. $-\frac{y}{x \log x}$.

14. ae^{ax}.

15. $4\,e^{4x+5}$.

16. $\frac{-e^{\frac{1}{1+x}}}{(1+x)^2}$.

17. $\frac{e^x}{(1+e^x)^2}$.

18. $y - 3\,x^2e^x$.

19. $1 - y^2$.

20. $\frac{e^x + e^{-x}}{e^x - e^{-x}}$.

21. $\frac{1+e^x}{x+e^x}$.

22. $nx^{n-1}a^x + x^na^x \log a$.

23. $\frac{\sqrt{a}}{\sqrt{x}(a-x)}$.

24. $-\frac{1}{x(\log x)^2}$.

25. $\frac{2 \log x}{x}$.

26. $\frac{1}{x \log x}$.

27. $-(\log x + 1)$.

28. $a^{\log x}\frac{\log a}{x}$.

29. $\frac{-(x-1)^{\frac{3}{2}}(7\,x^2 + 30\,x - 97)}{12(x-2)^{\frac{7}{4}}(x-3)^{\frac{10}{3}}}$.

30. $\frac{2 + x - 5\,x^2}{2\sqrt{1-x}}$.

31. $\frac{1 + 3\,x^2 - 2\,x^4}{(1-x^2)^{\frac{3}{2}}}$.

32. $5\,x^4(a+3\,x)^2(a-2\,x)$
$(a^2 + 2\,ax - 12\,x^2)$.

33. $\frac{(x-2\,a)\sqrt{x+a}}{(x-a)^{\frac{3}{2}}}$.

Page 41. Art. 22

1. $7 \cos 7\,x$.

2. $-5 \sin 5\,x$.

3. $2\,x \cos x^2$.

4. $2 \cos 2\,x \cos x - \sin 2\,x \sin x$.

5. $3 \sin^2 x \cos x$.

6. $10\,x \cos 5\,x^2$.

7. $14 \sin 7\,x \cos 7\,x$.

8. $\sec^2 x\,(\tan^2 x - 1)$.

9. $3 \sin^2 x \cos^2 x - \sin^4 x$.

10. $\sec x\,(\tan x + \sec x)$.

11. $-16\,x(1-2\,x^2) \sin (1-2\,x^2)^2$
$\cos (1-2\,x^2)^2$.

12. $-20\,x\,(3-5\,x^2)\sec^2(3-5\,x^2)^2$.

13. $2 \tan x \sec^2 x - 2 \tan x$.

14. $\sec x$.

15. $\frac{\cot \sqrt{x}}{2\sqrt{x}}$.

16. $-\frac{1}{x^2}\log a \cdot a^{\frac{1}{x}} \cdot \sec^2 (a^{\frac{1}{x}})$.

17. $n \sin^{n-1} x \sin (n+1)x$.

18. $\cos 2\,u \frac{du}{dx}$.

19. $\frac{mn \sin^{m-1} nx \cdot \cos(m-n)x}{\cos^{n+1} mx}$.

20. $\frac{2}{1 + \tan x}$.

21. $\cos(\sin u)\cos u\dfrac{du}{dx}$.

22. $2\,ae^{ax}\sin e^{ax}\cdot\cos e^{ax}$.

23. $e^x\cdot\cos e^x\cdot\log x+\dfrac{\sin e^x}{x}$.

24. $\dfrac{x\cos x^2}{\sqrt{\sin x^2}}$.

25. $-8\csc^2 4x\cot 4x$.

26. $8(4x-3)\sec(4x-3)^2$
$\tan(4x-3)^2$.

27. $-2x\csc^2 x^2+\dfrac{\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}}$.

28. $\dfrac{y\cos xy}{1-x\cos xy}$.

29. $-\csc^2(x+y)$.

Page 43. Art. 23

1. $\dfrac{4x}{\sqrt{1-4x^4}}$.

2. $\dfrac{1}{\sqrt{1-x^2}}$.

3. $\dfrac{3}{\sqrt{6x-9x^2}}$.

4. $\dfrac{3}{\sqrt{1-x^2}}$.

5. $\dfrac{-2}{1+x^2}$.

6. $\dfrac{1}{2\sqrt{\sin^{-1}x}\sqrt{1-x^2}}$.

7. $\dfrac{1}{e^x+e^{-x}}$.

8. $\dfrac{-1}{x\sqrt{1-(\log x)^2}}$.

9. $\dfrac{\sec^2 x}{\sqrt{1-\tan^2 x}}$.

10. $\dfrac{1}{\sqrt{1-x^2}}$.

11. $\dfrac{1}{\sqrt{1-x^2}}$.

12. $\dfrac{-1}{x\sqrt{x^2-1}}$.

13. $\dfrac{-1}{1+x^2}$.

14. $\frac{1}{2}\sqrt{1+\csc x}$.

15. $\frac{1}{2}$.

16. $\sec^2 x\cdot\tan^{-1}x+\dfrac{\tan x}{1+x^2}$.

17. $\sin^{-1}x+\dfrac{x}{\sqrt{1-x^2}}$.

18. $\dfrac{e^{\tan^{-1}x}}{1+x^2}$.

19. $\dfrac{2}{\sqrt{1-x^2}}$.

20. $\dfrac{-2}{x^2+1}$.

21. $\dfrac{1}{2\sqrt{x}(x+1)}$.

22. $\dfrac{-2}{e^x+e^{-x}}$.

23. $\dfrac{n}{\cos^2 x+n^2\sin^2 x}$.

24. 2.

25. $\dfrac{2\sin x}{\sqrt{1-4\cos^2 x}}$.

26. $\dfrac{-1}{2(1+x^2)}$.

27. $\dfrac{-1}{\sqrt{1-x^2}}$.

28. 0.

Page 45. Exercises on Chapter II

1. $6x+15x^2$.

2. $\dfrac{-6}{x^3}-\dfrac{15}{x^4}$.

3. $\dfrac{3x-1}{2\sqrt{x-3}}$.

4. $\dfrac{a^2-2x^2}{\sqrt{a^2-x^2}}$.

5. $\log\sin x+x\cot x$.

6. $\dfrac{-a^3}{x^2\sqrt{a^2-x^2}}$.

7. $\frac{c}{x}e^{\frac{x}{c}}\left(1-\frac{c}{x}\right)$.

8. $\frac{2x^2-2x+1}{2(x-x^2)^2}$.

9. $\frac{e^{\sqrt{u}}}{2\sqrt{u}}\cdot\cot x$.

10. $\frac{1}{x}-\log a$.

11. $\frac{-(3x+x^3)}{(1+x^2)^{\frac{3}{2}}}$.

12. $e^x(\cos x-\sin x)$.

13. $\frac{1}{x\sqrt{x^2-1}}$.

14. $\frac{4}{5+3\cos x}$.

15. $\tan^{-1}\sqrt{\frac{x}{a}}$.

16. $\frac{1}{2(1+x^2)}$.

17. $4\tan^5 x$.

18. $\frac{\log x}{(1-x)^2}$.

19. $\frac{4}{5+3\cos x}$.

20. $\frac{x^2}{1-x^4}$.

21. $\frac{1}{x}\sqrt{\frac{x+a}{x-a}}$.

22. 1.

23. $2\tan x+e^{\sec x}\cdot\sec x\tan x$.

24. $\frac{x^2-ay}{ax-y^2}$.

25. $-\frac{2xy^2+3x^2}{2x^2y+3y^2}$.

26. $\frac{3x^2+1}{3y^2+1}$.

27. $\frac{y^2+2xy-1}{1-2xy-x^2}$.

28. $\frac{4\cos(2\log x^2-7)}{x}$.

29. $x\frac{dy}{dx}=2y$.

30. $x=n\pi$.

32. x, y are determined from $a^2y=\pm b^2x$ and equation of curve.

33. $x=\kappa\pi\pm\frac{\pi}{4}$.

34. $\tan^{-1}2\sqrt{2}$.

Pages 49, 50. Exercises on Chapter III

1. $72x$.

2. 0.

3. $-\frac{3!}{x^4}$.

4. $-\frac{5!}{x^6}$.

5. $6\sec^4 x-4\sec^2 x$.

6. $e^x\log x+\frac{2e^x}{x}-\frac{e^x}{x^2}$.

7. $2\log x+3$.

8. $8\tan x\sec^2 x(3\sec^2 x-1)$.

9. $2\cot x\csc^2 x$.

10. $16\sin x\cos x$.

11. $\frac{24}{(1-x)^5}$.

12. $\frac{48}{x}$.

13. $\sin x$.

14. $-\frac{8(e^x-e^{-x})}{(e^x+e^{-x})^3}$.

15. $8x^2e^{2x}$.

16. $-\dfrac{4!}{x^2}$.

17. a^ne^{ax}.

18. $\dfrac{(-1)^n n!}{(x-1)^{n+1}}$.

19. $m^n \cdot \cos\left(mx + n\dfrac{\pi}{2}\right)$.

20. $\dfrac{(-1)^n \cdot (m+n-1)!}{(m-1)!\,(a+x)^{m+n}}$.

21. $\dfrac{m(-1)^{n-1} \cdot (n-1)!}{(a+x)^n}$.

22. $\dfrac{3p^3}{y^5}$.

23. $\dfrac{-b^4}{a^2y^3}$.

24. $\dfrac{-2a^3xy}{(y^2-ax)^3}$.

25. $\dfrac{-y[(x-1)^2+(y-1)^2]}{x^2(y-1)^3}$.

26. $\dfrac{3-y}{(2-y)^3} \cdot e^{2y}$.

32. $\dfrac{(n-1)!}{x}$.

34. $2^{n-1}\cos\left(2x + \dfrac{n\pi}{2}\right)$.

Page 53. Art. 28

2. Inc. from $-\infty$ to $\frac{1}{3}$; dec. from $\frac{1}{3}$ to 1; inc. from 1 to $+\infty$; $\frac{1}{3}$ and 1.

3. Two. $+1$ at $x = \frac{1}{2} \pm \sqrt{\frac{5}{12}}$; -1 at $x = \frac{1}{2} \pm \sqrt{\frac{1}{12}}$. 4. $\pm \tan^{-1}\frac{4}{97}$.

Page 60. Art. 34

1. $-\dfrac{1}{\sqrt{3}}$, max.; $\dfrac{1}{\sqrt{3}}$, min.

2. 2, max.; 3, min.

3. 2, min.; $\frac{8}{5}$, max.

4. $(2n+\frac{1}{4})\pi$, max.; $(2n+\frac{5}{4})\pi$, min. for all integral values of n.

5. $\dfrac{a}{4}$, min.

6. -1, max.; $-\frac{1}{3}$, min.

7. -2, min.; 1, max.

8. e, max.

9. $2n\pi$, min.; also $\tan^{-1} \pm \sqrt{\frac{2}{3}}$ for angles in 2d and 3d quarter. $(2n+1)\pi$, $\tan^{-1} \pm \sqrt{\frac{2}{3}}$, 1st and 4th quarter, max.

10. 2, min.; -1, max.

11. $x = \dfrac{3u+4}{2(u-1)}$, $u^2 + 44m + 4 = 0$.

Pages 63–67. Exercises on Chapter IV

1. Two thirds the length of the segment.

2. The parts are equal.

3. $\frac{1}{2}$.

4. 1.

5. $\dfrac{a}{\sqrt{3}}$.

6. 3 inches.

7. $\dfrac{2r}{\sqrt{3}}$.

8. Area is $\dfrac{ab}{2}$.

9. The side parallel to the wall is double each of the others.

10. The altitude is equal to the diameter of the base.

11. 8 inches.

12. One mile from stopping point.

13. Most economical per hour at 15 knots.

14. $\frac{3}{2}a$.

15. The altitude of the rectangle is equal to the radius.

16. The altitude is equal to the radius of the base.

17. $\dfrac{20}{\sqrt{1-\dfrac{1}{4^2 \cdot 12^2}}}$ yards from the nearest point.

19. $15\sqrt{2}$ feet.

20. The diameter of the sphere equals the edge of the cube.

21. $\dfrac{50}{\sqrt{2}}$ feet.

22. Circular arc is double the radius.

23. $\dfrac{Dr^{\frac{3}{2}}}{r^{\frac{3}{2}}+R^{\frac{3}{2}}}$, D being the distance between the centers of the spheres.

24. $\text{Arc} = 2\pi r(1-\sqrt{\frac{2}{3}})$.

25. Angle at center of variable circle defined by $\theta = \cot\theta$.

26. The line should be bisected at the given point.

27. The altitude is $\frac{3}{4}$ the slant height of the cone.

28. $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$.

29. $\frac{3}{4}a$.

30. $a+b$.

31. $x = a - p$.

32. 20 ft.

33. $a\sqrt{\frac{2}{3}}$.

34. $x = \sqrt{\dfrac{2aW}{w}}$ feet.

35. $\tan\theta = \sec\phi - \tan\phi$.

36. $\theta = 35° 20'$.

Pages 76, 77. Art. 39

3. About 3° 58′ per second.

4. 120 feet per minute.

5. $(3, \frac{16}{3})$.

6. At $5\sqrt{2}$ miles per hour.

7. (3, 6).

8. At 60°.

9. $2ab$.

10. ± 2.

11. 5π.

12. 2.

13. 1 and 5.

16. $s = \frac{v_0^2}{64}, \; t = \frac{v_0}{32}.$

17. $\frac{a}{16}.$

19. $\pm 16, \; \mp 12$ feet per second.

20. $\sin \phi \cdot d\phi.$

21. $\frac{36}{\sqrt{37^3}}$ radians per second.

Page 83. Exercises on Chapter VI

1. $\sqrt{\frac{a+x}{x}}, \; 2\sqrt{ax}, \; 4\pi\sqrt{a^2 + ax}, \; 4\pi ax.$

2. $\frac{a}{y}, \; \frac{a}{x}.$

3. $\sec x.$

4. $\pi a^2 x^2.$

5. $\pi \frac{b^2}{a^2}(a^2 - x^2).$

6. $\rho\sqrt{1 + (\log a)^2}.$

7. $\sqrt{2 a\rho}.$

8. $\frac{a^2}{\rho}.$

9. $30\pi, \; 72\pi.$

10. $\frac{2}{3}\sin\theta$, $90°$, $270°$; 2, -2.

Pages 87–89. Art. 48

1. $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1,$

$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1).$

2. $y = x.$

3. $2y = 9x - 3, \; 9y + 2x = 29.$

4. (α) $x + 2y = 4a,$
$y - 2x + 3a = 0.$
(β) $2y = \pm(x + 1),$
$y = \mp 2x \pm 3.$
(γ) $y = x + p, \; x + y - 3p = 0.$

5. 3.

6. $4\sqrt{17}.$

7. (α) Parallel at points of intersection with $ax + hy = 0$.
Perpendicular at points of intersection with $hx + by = 0$.

(β) Parallel at $\left(\frac{a}{-\sqrt[3]{2}}, \; \frac{3a\sqrt[3]{2}}{2}\right)$; perpendicular at $x = 0$.

(γ) Parallel at $\left(\frac{4a}{3}, \; \frac{2\sqrt[3]{4}\,a}{3}\right)$; perpendicular at (0, 0); $(2a, 0)$.

8. $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$, *i.e.* they must be confocal.

9. $\frac{\pi}{2}.$

11. $\frac{\pi}{2}.$

12. $\frac{y^2}{nx}.$

13. $\frac{2c + a}{3}.$

19. $(2p, \; \pm 2p\sqrt{2}).$

Page 95. Art. 51

1. An inflexion at $x = y = 2$.

3. $\left(\frac{2a}{\sqrt{3}}, \; \frac{3a}{2}\right); \; \left(\frac{-2a}{\sqrt{3}}, \; \frac{3a}{2}\right).$

6. $x - 4a^2 y = 0.$

8. Point of inflexion at $(a, \frac{4}{3}a)$, tangent is $x + y = \frac{7a}{3}$. Bending changes from negative to positive.

10. $(-1, 1)$, $\left(2 \pm \sqrt{3}, \frac{-(1 \pm \sqrt{3})}{4(2 \pm \sqrt{3})}\right)$.

Page 103. Art. 57

1. $y = 0$, $x = a$, $x = -a$.
2. $x = 0$, $x = 2a$, $y = a$, $y = -a$.
3. $y = a$, $y = -a$; two imaginary.
4. $y = a$; $x = c$ twice.
5. $y = -x + \frac{a}{3}$; two imaginary.
6. $x = 1$; one parabolic branch.
7. $x = -a$, $y = -b$, $y = x + b - a$.
8. $x = 0$ twice; one parabolic branch.
9. $x = 0$, $y = 0$, $x + y = 0$.
10. $y = x$; two imaginary.
11. $x + y + a = 0$; two imaginary.
12. $y + x = 0$; two imaginary.
13. $x = 0$ twice; $x = y$, $x = -y$.
14. $y = x$, $y = -x$; two imaginary.
15. $x + 2y = 0$, $x + y = 1$, $x - y = -1$.

Page 107. Art. 60

1. $\psi = \theta$.
2. Polar subtangent $= \frac{\rho^2}{a}$, Polar normal $= \sqrt{a^2 + \rho^2}$, Polar subnormal $= a$.
3. $\psi = \frac{\pi}{2} + 2\theta$, Subtangent $= -\rho \cot 2\theta$, Tangent $= \frac{a^2 \rho}{\sqrt{a^4 - \rho^4}}$,

   Subnormal $= -\frac{a^2 \sin 2\theta}{\rho}$, Normal $= \frac{a^2}{\rho}$.
5. $\frac{\theta}{2}$, $2a \sin^2 \frac{\theta}{2} \cdot \tan \frac{\theta}{2}$.
7. They have a common tangent at the pole; elsewhere, $\frac{\pi}{2}$.

Page 111. Art. 62

3. 1.
4. $(x + y) \cos xy$.
5. 1.

Page 115. Art 63

5. $\frac{7}{2}$ square units.
6. $5\sqrt{10}$.
7. Differs by $dx\,dy$.

Page 117 Art. 65

3. $-\dfrac{ax+hy+g}{hx+by+f}$.

4. $\dfrac{x^3}{y^3}$.

6. $\dfrac{2v}{s}+vy+\dfrac{y}{s}$.

7. $\dfrac{3}{\sqrt{1-t^2}}$.

8. $\dfrac{e^x-y}{e^y+x}$.

9. $\dfrac{y[\cos(xy)-e^{xy}-2x]}{x[x+e^{xy}-\cos(xy)]}$.

Page 121. Art. 66

2. $8x+8y-z-12=0$.

3. $\dfrac{x-2}{1}=\dfrac{y-2}{-4}=\dfrac{z-3}{3}$.

5. $\dfrac{x}{x_1^{\frac{1}{3}}}+\dfrac{y}{y_1^{\frac{1}{3}}}+\dfrac{z}{z_1^{\frac{1}{3}}}=a^{\frac{2}{3}}$.

7. $\cos^{-1}\dfrac{15}{\sqrt{119}}$.

8. $\dfrac{9}{\sqrt{17}}$.

9. $2x_1x+y_1y-z_1z=0$; $\dfrac{x-x_1}{2x_1}=\dfrac{y-y_1}{y_1}=\dfrac{z-z_1}{-z_1}$.

10. $2x+2\sqrt{3}\,y+3z=25$, $x+z=5$.

Pages 128–130. Art. 72

2. $\dfrac{d^2x}{dy^2}-2y\dfrac{dx}{dy}=0$.

3. $R=-\dfrac{\left\{1+\left(\dfrac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}}$.

4. $\left(\dfrac{d^3x}{dy^3}+\dfrac{d^2x}{dy^2}\right)\dfrac{dx}{dy}=0$.

5. $\dfrac{d^2z}{dx^2}=\cos^2 z+2\left(\dfrac{dz}{dx}\right)^2$.

6. $\dfrac{d^2u}{dy^2}+u=0$.

8. $\dfrac{d^2y}{du^2}+y=0$.

9. $\dfrac{d^2y}{dt^2}=0$.

11. $\dfrac{d^2x}{dy^2}+\dfrac{dx}{dy}+y=0$.

12. $2z\dfrac{d^2z}{dx^2}+2\left(\dfrac{dz}{dx}\right)^2$
$+(1-z^2)\,2z\dfrac{dz}{dx}+z^4=0$.

13. $\dfrac{d^2y}{dt^2}+y=0$.

14. $\dfrac{d^3v}{dt^3}+v=0$.

15. $\dfrac{d^2y}{dt^2}+\dfrac{2}{t}\dfrac{dy}{dt}+a^2y=0$.

16. $\dfrac{\rho^2}{\sqrt{\rho^2+\left(\dfrac{d\rho}{d\theta}\right)^2}}$.

17. -6.

Page 137. Art. 74

6. Divergent. 7. Convergent. 8. Convergent in both cases.

Page 140. Art. 75

6. $-1 < x < 1$. 7. $|x| > 1$. 8. $-a < x < a$.

Page 145. Art. 77

2. $$f(x) = (x-1)^3 + (x-1)^2 + 4(x-1) - 3.$$
$$f(1.02) = -2.919592,\quad f(1.01) = -2.959899.$$
$$f(.99) = -3.038901,\quad f(.98) = -3.079608.$$

3. $3(y-3)^2 + 4(y-3) - 8$. 4. $\sin 31° = .51503$.

Page 149. Art. 78

1. $x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + R$. 3. .000002.

6. .017452. 7. $1 - \dfrac{x^2}{2} - \dfrac{x^4}{8} + R$.

8. $\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}\left(x - \dfrac{\pi}{6}\right) - \dfrac{\sqrt{3}}{4}\left(x - \dfrac{\pi}{6}\right)^2 + \dfrac{1}{12}\left(x - \dfrac{\pi}{6}\right)^3 + R$.

9. $e^x + e^x h + \dfrac{e^x}{2\,!}h^2 + R$.

10. $15 + 24(x-2) + 13(x-2)^2 + 3(x-2)^3$.

11. $\log x + \dfrac{h}{x} - \dfrac{h^2}{2x^2} + \dfrac{h^3}{3x^3} - \dfrac{h^4}{4x^4} + R$.

12. $-4(x+1) + 6(x+1)^2 - 4(x+1)^3 + (x+1)^4$.

14. 5.013. 16. 3.433987.

15. 11.0087. 17. $.0127 \cdots$.

18. $1 - (x-1) + (x-1)^2 - (x-1)^3 + R$. 0 to 2.

Pages 159, 160. Art. 83

2. $\dfrac{2a^2}{a^2+b^2}$. 3. $-\dfrac{13}{7}$. 4. $\dfrac{1}{2a}$.

Pages 163, 164. Art. 85

3. $\frac{2}{3}$. 5. $\frac{2}{5}$. 8. -4.

4. 4. 7. $\frac{1}{4}$.

Page 164. Art. 85

1. 0.
2. 2.
3. 3.
4. $\dfrac{\log a}{\log b}$.
5. $\dfrac{a}{b}$.
6. $\dfrac{1}{n}$.
7. 1.
8. 1.
9. 1.
10. $-\dfrac{2}{3}$.
11. $\frac{1}{2}$.

Page 166. Art. 86

1. 1.
2. 0.
3. 0.
4. 5.
5. $-\frac{5}{3}$.
6. 1.
7. $e^{-\frac{a^2}{2c^2}}$.

Page 171. Art. 89

1. First.
2. They do not touch.
3. Third.
4. $3x(x-a)=a(y-a)$.
5. $y+12x=10$.
6. Second.
7. $a=-1$.
8. $y=2x^2-5x+4$.

9 *a*. $(-2, -8)$, First.

9 *b*. $(-2, -8)$, Second.

10. $y=-x^2+2x+3$, $8x=-3y^2+14y-36$.

12. First.
13. Second.

Page 179. Art. 97

1. $2a$.
2. $\dfrac{a}{2}$.
3. ∞.
4. $\dfrac{(x^2+y^2)^{\frac{3}{2}}}{2m^2}$.
5. $\dfrac{(e^2x^2-a^2)^{\frac{3}{2}}}{ab}$.
6. $\dfrac{(x^2+n^2y^2)^{\frac{3}{2}}}{n(n-1)xy}$.
7. $\dfrac{2(x+y)^{\frac{3}{2}}}{\sqrt{a}}$.
8. $3(axy)^{\frac{1}{3}}$.
9. $\dfrac{a\sqrt{x(8a-3x)^3}}{3(2a-x)^2}$.
10. $\dfrac{y^2}{a}$.
11. $\left(\dfrac{1}{3\sqrt{2}}, \dfrac{1}{54\sqrt{2}}\right)$.

Page 181. Art. 98

1. $\rho\sqrt{1+(\log a)^2}$.
2. $\dfrac{a^2}{3\rho}$.
3. $\dfrac{a(5-4\cos\theta)^{\frac{3}{2}}}{9-6\cos\theta}$.

4. $\dfrac{2\rho^{\frac{3}{2}}}{\sqrt{a}}$.

5. $-\dfrac{\rho^3}{a^2}$.

6. $\dfrac{4\sqrt{\rho a}}{3}$.

7. $\dfrac{a(1+\theta^2)^{\frac{3}{2}}}{\theta^4}$.

Page 188. Art. 100

1. $\alpha = 0,\ \beta = 0.$

2. $\alpha = a\log\dfrac{a+\sqrt{a^2-y^2}}{y},\ \beta = \dfrac{a^2}{y}.$

3. $\alpha = \dfrac{a^4+15\,y^4}{6\,a^2y},\ \beta = \dfrac{a^4y-9\,y^5}{2\,a^4}.$

4. $\alpha = x - \dfrac{y}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right),\ \beta = 2\,y.$

5. $(\alpha+\beta)^{\frac{2}{3}} - (\alpha-\beta)^{\frac{2}{3}} = (4\,a)^{\frac{2}{3}}.$

6. $(a\alpha)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}.$

7. $(\alpha+\beta)^{\frac{2}{3}} + (\alpha-\beta)^{\frac{2}{3}} = 2\,a^{\frac{2}{3}}.$

16. $\alpha = a(\theta' - \sin\theta'),\ \beta = a(1-\cos\theta'),\ \theta' = \theta - \pi.$

Pages 198, 199. Exercises on Chapter XIII

1. $(0, 0)$; $ax \pm by = 0$.

2. $(0, 0)$; cusp of first kind, $y=0$.

3. Four cusps of first kind; $(0, \pm a)$, $(\pm a, 0)$; $y = 0$, $x = 0$.

4. $(0, 0)$; conjugate point with real coincident tangents, $y = 0$.

5. $(0, a)$; $y = a + x$; cusp of second kind.

6. Two nodes at infinity; the asymptotes are $x = y \pm 1$, $x + y = \pm 1$.

7. $(0, -a)$; $(+a, 0)$; $(-a, 0)$ the tangents are, respectively,

$$\sqrt{3}(y+a) = \pm\sqrt{2}\,x;$$
$$2(x+a) = \pm\sqrt{3}\,y;$$
$$2(x-a) = \pm\sqrt{3}\,y.$$

8. $(-a, 0)$; conjugate points.

9. $(0, 0)$; $x = 0$, $y = 0$.

10. $(0, 0)$; is a tacnode; $y = 0$.

12. Terminating point at $(0, 0)$.

Pages 207, 208. Exercises on Chapter XIV

1. $x^2 + y^2 = p^2.$

2. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

3. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$

4. $4\,xy = k^2.$

5. $(x-\alpha)^2 + (y-\beta)^2 = r^2.$

6. $y^2(x+2\,a) + x^3 = 0.$

7. $y^2 = 4\,a(2\,a - x).$

8. $b^2x^2 + (a^2+b^2)y^2 = b^2(a^2+b^2).$

9. $(x^2+y^2)^2 = 4\,c^2(x^2-y^2).$

10. $16\,y^3 + 27\,x^4 = 0.$

11. $y \pm x \pm k = 0.$

12. $(x^2+y^2-ay)^2 = a^2(x^2+(y+a)^2).$

13. $b^2x^2 + a^2y^2 = 4\,a^2b^2.$

INTEGRAL CALCULUS

Pages 215–216. Art. 114

1. $\frac{2}{3} x^{\frac{3}{2}}$.
2. $\dfrac{x^{a+1}}{a+1}$.
3. $\frac{3}{2} x^{\frac{2}{3}}$.
4. $\dfrac{2m}{2m-1} x^{m+\frac{1}{2}}$.
5. $ax - \frac{9}{4} a^{\frac{2}{3}} x^{\frac{4}{3}} + \frac{9}{5} a^{\frac{1}{3}} x^{\frac{5}{3}} - \frac{1}{2} x^2$.
6. $5 \log x + \dfrac{3}{2x^2} - \dfrac{1}{3x^3}$.
7. $\frac{1}{6}(x^2 + a^2)^3$.
8. $\dfrac{(ax+b)^{n+1}}{a(n+1)}$.
9. $\log (x + a)$.
10. $\frac{1}{2} \log (2ax - x^2)$.
11. $-\log \cot x$.
12. $-\log (1 + \cos x)$.
13. $\log (\log x)$.
14. $\frac{5}{3} \log (x^3 + 1)$.
15. $-\log \cos x$.
16. $\log \sin x$.
17. $\dfrac{1}{a} e^{ax}$.
18. $\frac{1}{2} e^{x^2}$.
19. $\dfrac{(a+b)^{m+nx}}{n \log (a+b)}$.
20. $\frac{1}{2} \sin 2x$.
21. $-\dfrac{1}{n} \cos nx$.
22. $\dfrac{x}{2} + \dfrac{\sin 2x}{4}$.
23. $\dfrac{x}{2} - \dfrac{\sin 2x}{4}$.
24. $-\dfrac{\cos(m+n)x}{m+n}$.
25. $-\frac{1}{2} \cos x^2$.
26. $\sin x - \frac{1}{3} \sin^3 x$.
27. $-\cos x + \frac{1}{3} \cos^3 x$.
28. $\tan x - x$.
29. $\frac{1}{3} \tan^3 x$.
30. $-\dfrac{1}{a} \cot (ax + b)$.
31. $-\frac{2}{3}(\cot x)^{\frac{3}{2}}$.
32. $\log \tan x$.
33. $\frac{1}{3} \sec^3 x$.
34. $-\cos x$.
35. $\sin^{-1} \dfrac{x}{a}$.
36. $\frac{1}{2} \sin^{-1} 2x$.
37. $\dfrac{1}{a} \tan^{-1} \dfrac{u}{a}$.
38. $\dfrac{1}{ab} \tan^{-1} \dfrac{ax}{b}$.
39. $\tan^{-1} (x - 2)$.

Page 219. Art. 115

1. $x \sin^{-1} x + \sqrt{1 - x^2}$.
2. $e^x \tan^{-1} e^x - \frac{1}{2} \log (1 + e^{2x})$.
3. $x^2 \sin x + 2x \cos x - 2 \sin x$.
4. $\dfrac{x^{n+1}}{n+1}\left(\log x - \dfrac{1}{n+1}\right)$.

5. $\frac{1}{6}[2x^3 \tan^{-1} x - x^2 + \log(1+x^2)]$.

6. $\sec x\,[\log \cos x + 1]$.

7. $\frac{1}{2}[(x^2+1)\cot^{-1} x + x]$.

8. $\frac{1}{9}[\sin 3x - 3x \cos 3x]$.

9. $\frac{1}{2} e^x(\sin x + \cos x)$.

10. $\frac{1}{2} e^x(\sin x - \cos x)$.

11. $\frac{2}{3} \cos x \sin 2x - \frac{1}{3} \cos 2x \sin x$.

12. $x \tan x + \log \cos x$.

Pages 220–222. Art. 116

4. $\frac{1}{2}(\sin^{-1} x)^2$.

6. $\frac{1}{2}\cos(x^2+1)[1-\log\cos(x^2+1)]$.

7. $\sin^{-1} \dfrac{u}{a}$.

8. $\dfrac{1}{a} \tan^{-1} \dfrac{u}{a}$.

9. $\dfrac{1}{a} \cos^{-1} \dfrac{a}{x}$.

10. $\sin^{-1} \dfrac{x-a}{a}$.

11. $\log \tan \dfrac{u}{2}$.

12. $\log \tan \left(\dfrac{u}{2} + \dfrac{\pi}{4}\right)$.

13. $-\frac{2}{9}(a^3 - x^3)^{\frac{3}{2}}$.

14. $\log(x-1) - \dfrac{2}{x-1} - \dfrac{1}{2(x-1)^2}$

15. $-\dfrac{1}{2\sin^2 x}$.

16. $\dfrac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x)$.

Page 226. Art. 118

1. $\dfrac{1}{2\sqrt{2}} \log \dfrac{\sqrt{2}(x+1)-1}{\sqrt{2}(x+1)+1}$.

2. $\dfrac{1}{\sqrt{14}} \tan^{-1} \dfrac{3x-1}{\sqrt{14}}$.

3. $-\dfrac{1}{12} \log \dfrac{x-2}{x+1}$.

4. $\frac{1}{3} \sin^{-1}(3x-5)$.

5. $\sqrt{x^2+2x+2} - \log(x+1+\sqrt{x^2+2x+2})$.

6. $-\sqrt{-x^2+2x+1} + \sin^{-1} \dfrac{x-1}{\sqrt{2}}$.

7. $-\sqrt{8-4x-4x^2} + \frac{3}{2} \sin^{-1} \dfrac{2x+1}{3}$.

8. $\dfrac{2}{3}\sqrt{3x^2+x-2} - \dfrac{10}{3\sqrt{3}} \log(x + \frac{1}{6} + \sqrt{x^2 + \frac{1}{3}x - \frac{2}{3}})$.

9. $\dfrac{1}{2}\sqrt{1+x-2x^2} + \dfrac{3}{4\sqrt{2}} \sin^{-1} \dfrac{4x-1}{3}$.

10. $\frac{17}{4} \log(x-5) - \frac{5}{4} \log(x-1)$.

11. $\sqrt{ax-x^2} + \dfrac{a}{2} \sin^{-1} \dfrac{2x-a}{a}$.

12. $-\sqrt{-2x^2-3x-1}-\dfrac{1}{2\sqrt{2}}\sin^{-1}(4x+3)$.

13. $-\frac{1}{3}\sqrt{1-2x-3x^2}-\dfrac{10}{3\sqrt{3}}\sin^{-1}\dfrac{3x+1}{2}$.

Page 227. Art. 119

1. $-\dfrac{1}{a}\log\left(\dfrac{a+\sqrt{x^2+a^2}}{x}\right)$.

2. $-\dfrac{1}{a}\log\left(\dfrac{a+\sqrt{a^2-x^2}}{x}\right)$.

3. $-\log\left(\dfrac{1-2x+\sqrt{5x^2-4x+1}}{x}\right)$.

4. $-\dfrac{1}{\sqrt{2}}\log\left(\dfrac{\sqrt{2}+\sqrt{x^2+2x+3}}{x+1}\right)$.

5. $-\log\left(\dfrac{1-x+2\sqrt{x^2+x+1}}{x+1}\right)$.

6. $\sin^{-1}\dfrac{1}{\sqrt{2}(x-1)}$.

7. $-\dfrac{1}{3}\log\left(\dfrac{1-x+\sqrt{-x^2-10x-7}}{x+2}\right)$.

8. $\dfrac{-\sqrt{a^2-x^2}}{a^2x}$.

9. $\dfrac{\sqrt{x^2-a^2}}{a^2x}$.

10. $-\dfrac{\sqrt{x^2+a^2}}{a^2x}$.

11. $-\dfrac{1}{\sqrt{2}}\sin^{-1}\dfrac{3-2x}{\sqrt{3}(2x-1)}$.

Pages 227–228. Exercises on Chapter I

1. e^{e^x}.

2. $\dfrac{5}{8}\tan^{-1}\dfrac{x^4}{2}$.

3. $\frac{1}{6}\log(6x^3+12x+5)$.

4. $2\sqrt{x}(1+\frac{1}{3}x)$.

5. $-\frac{3}{4}(3-2x)^{\frac{2}{3}}$.

6. $\dfrac{1}{2b(a-bx)^2}$.

7. $-\frac{3}{8}(a^2-x^2)^{\frac{4}{3}}$.

8. $\dfrac{-1}{\sqrt{x^2+1}}$.

9. $\frac{1}{3}(x+1)^{\frac{3}{2}}-\frac{1}{3}(x-1)^{\frac{3}{2}}$.

10. $\frac{1}{8}\sin 8x$.

11. $\dfrac{1}{3}\log\tan\left(\dfrac{3x}{2}+\dfrac{\pi}{4}\right)$.

12. $-\cos e^x$.

13. $-\dfrac{1}{a}\log(a\cos x+b)$.

14. $-\log(e^{-x}+\sqrt{e^{-2x}-1})$.

15. $\frac{1}{2}\sin^{-1}x^2$.

16. $-\dfrac{1}{2\sqrt{2}}\log\left[\dfrac{\sqrt{2}+\sqrt{x^2+2}}{x}\right]$.

17. $\dfrac{1}{2}\log\dfrac{e^x-1}{e^x+1}$.

18. $\frac{1}{5}x^5\tan^{-1}x-\frac{1}{20}x^4+\frac{1}{10}x^2-\frac{1}{10}\log(x^2+1)$.

19. $\dfrac{-[2+2x\log a+(x\log a)^2]}{a^x(\log a)^3}$. **20.** $\tan\theta-\sec\theta$.

21. $-\cot\dfrac{\theta}{2}$. **22.** $\dfrac{\log(a\cos^2\theta+b\sin^2\theta)}{2(b-a)}$.

23. $-\frac{1}{3}\sqrt{1-\log x}$. **24.** $\frac{1}{2}\log(e^{2x}+1)$. **25.** $\sin^{-1}\left(\dfrac{2\sin x+1}{3}\right)$.

26. $\tan^{-1}(\log x)$. **27.** $\dfrac{1}{b(a-b\tan x)}$.

28. $\dfrac{1}{2}\sin^{-1}\left[\dfrac{2(x-a)^2+a^2}{\sqrt{5}\,a^2}\right]$.

29. $-\dfrac{1}{x}\sqrt{3x^2+2x+1}+\log\left[\dfrac{x+1+\sqrt{3x^2+2x+1}}{x}\right]$.

30. $-\dfrac{1}{2x^2}\log(x+\sqrt{x^2-a^2})+\dfrac{\sqrt{x^2-a^2}}{2a^2x}$.

31. $-\cos x\log\tan x+\log\tan\dfrac{x}{2}$. **32.** $\frac{1}{2}x-\frac{1}{2}\log(\sin x+\cos x)$.

Pages 236–237. Exercises on Chapter II

1. $\dfrac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2}+\dfrac{3a^4}{8}\sin^{-1}\dfrac{x}{a}$.

2. $\dfrac{1}{8}\left[\dfrac{x}{x^2+4}+\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}\right]$. 5. $\dfrac{x}{2}\sqrt{a^2-x^2}+\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$.

3. $\dfrac{2x-1}{3(x^2-x+1)}+\dfrac{4}{3\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}$. 6. $-\dfrac{\sqrt{a^2-x^2}}{a^2x}$.

4. $-\dfrac{x}{2}\sqrt{a^2-x^2}+\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$. 7. $\dfrac{x}{3a(x^2+a)^{\frac{3}{2}}}+\dfrac{2x}{3a^2\sqrt{x^2+a}}$.

8. $\dfrac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2}+\dfrac{3a^4}{8}\log(x+\sqrt{x^2+a^2})$.

9. $\dfrac{x}{2}\sqrt{x^2+a}+\dfrac{a}{2}\log(x+\sqrt{x^2+a})$.

10. $\dfrac{1}{6}(2x^2-ax-3a^2)\sqrt{2ax-x^2}+\dfrac{a^3}{2}\sin^{-1}\dfrac{x-a}{a}$.

11. $-\dfrac{(2ax-x^2)^{\frac{3}{2}}}{3ax^3}$. 12. $-\dfrac{\sqrt{1+x^4}}{2x^2}$.

13. $\dfrac{3(x+2)^3-5(x+2)}{8(x^2+4x+3)^2}+\dfrac{3}{16}\log\dfrac{x+1}{x+3}$.

14. $\frac{1}{2}(x+1)\sqrt{1-2x-x^2}+\sin^{-1}\frac{x+1}{\sqrt{2}}$.

16. $\frac{\sqrt{x^2-1}}{2x^2}-\frac{1}{2}\sin^{-1}\frac{1}{x}$.

18. $\frac{x}{a\sqrt{a+bx^2}}$.

17. $\frac{-1}{2(x^2+7)}$.

19. $\frac{-\cos\theta}{(1+e)\sqrt{1+e\sin^2\theta}}$.

20. $\frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2}+\frac{a^4}{8}\sin^{-1}\frac{x}{a}$.

21. $\frac{x}{48}(33a^4-26a^2x^2+8x^4)\sqrt{a^2-x^2}+\frac{5a^6}{16}\sin^{-1}\frac{x}{a}$.

Page 241. Art. 122

1. $\frac{1}{2a}\log\frac{x-a}{x+a}$.

3. $\frac{x^2}{2}-4x+\frac{13}{2}\log\frac{(x+3)^3}{x+1}$.

2. $\log\frac{(x+1)^2}{x(x-1)}$.

4. $x+\log(x-a)^a(x-b)^b$.

5. $\frac{2+\sqrt{3}}{2\sqrt{3}}\log(x-2-\sqrt{3})-\frac{2-\sqrt{3}}{2\sqrt{3}}\log(x-2+\sqrt{3})$.

6. $\frac{1}{20}\log\frac{(2x-1)(x-2)}{(2x+1)(x+2)}$.

7. $\log\frac{(x-a)(x-b)}{x-c}$.

8. $x+\frac{1}{b-a}[a^2\log(x+a)-b^2\log(x+b)]$.

9. $\log[(x+2)\sqrt{2x-1}]$.

12. $\frac{x^2}{2}-7x+64\log(x+4)$
$-27\log(x+3)$.

10. $\log\frac{(x-a)(x+b)}{x}$.

13. $\frac{1}{2ab}\log\frac{ax-b}{ax+b}$.

11. $\frac{1}{3}\log\frac{x^6}{(2+x)(1-x)^5}$.

14. $\frac{1}{2}\log\frac{1+t}{1-t}$.

Page 243. Art. 123

1. $-\frac{1}{2(x-1)}+\frac{1}{4}\log\frac{x+1}{x-1}$.

3. $\frac{1}{2(a^2-x^2)}$.

2. $\frac{1+2x}{2x^2}+\log\frac{x-1}{x}$.

4. $\frac{-1}{\sqrt{2}x(x+\sqrt{2})}$.

5. $\frac{x^2}{2}-2x+\frac{2x+3}{x^2+x}+\log x(x+1)^2$.

6. $\log(x^2-a^2)-\dfrac{2a^2}{x^2-a^2}$.

7. $\dfrac{1}{4(\sqrt{2}+1-x)^2}$.

8. $ax-\dfrac{1}{x}+\log\dfrac{x}{x+a}$.

9. $x+\dfrac{1}{3x}-\dfrac{1}{9}[28\log(x+3)-\log x]$.

10. $\dfrac{b}{a^2}\log\dfrac{a+bx}{x}-\dfrac{1}{ax}$.

11. $x-\dfrac{2}{x-3}-5\log(x-3)$.

12. $\log(x-a)+\dfrac{3a^2-4ax}{2(x-a)^2}$.

Page 244. Art. 124

1. $\log\dfrac{x}{\sqrt{x^2+4}}$.

2. $\dfrac{1}{4}\log\dfrac{x^2+1}{(x+1)^2}+\dfrac{1}{2}\tan^{-1}x$.

3. $\dfrac{1}{3a^2}\left[\log(x+a)-\dfrac{1}{2}\log(x^2-ax+a^2)+\sqrt{3}\tan^{-1}\dfrac{2x-a}{a\sqrt{3}}\right]$.

4. $-\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+\dfrac{1}{b}\tan^{-1}\dfrac{x}{b}$.

7. $-\dfrac{1}{2a(x-a)}-\dfrac{1}{2a^2}\tan^{-1}\dfrac{x}{a}$.

5. $\dfrac{3}{x}+\log\dfrac{x^2}{x^2+2}+\dfrac{3}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}}$.

8. $x-\log\dfrac{x^2+2x+2}{x-1}$.

6. $-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{\sqrt{3}}{2x^2+1}$.

9. $\log\dfrac{x-1}{\sqrt{x^2+1}}-\tan^{-1}x$.

Page 246. Art. 125

1. $\tan^{-1}x+\dfrac{1}{x^2+1}$.

2. $\dfrac{3}{2a}\tan^{-1}\dfrac{x}{a}+\dfrac{x-2a}{2(x^2+a^2)}$.

3. $\dfrac{1}{4}\log\dfrac{x^2+1}{(x+1)^2}+\dfrac{x-1}{2(x^2+1)}$.

4. $-\dfrac{2}{x}+3\log\dfrac{\sqrt{x^2+1}}{x}-\dfrac{3+2x}{2(x^2+1)}-3\tan^{-1}x$.

5. $\dfrac{x}{2(x^2+a^2)}+\log(x^2+a^2)-\dfrac{1}{2a}\tan^{-1}\dfrac{x}{a}$.

6. $\dfrac{1}{x^2+1}-\dfrac{1}{4(x^2+1)^2}+\dfrac{1}{2}\log(x^2+1)$.

Page 249. Art. 127

1. $\log \dfrac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$.

2. $2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\log(\sqrt[6]{x}+1)$.

3. $2\log(\sqrt{x-1}+1) + \dfrac{2}{\sqrt{x-1}+1}$.

4. $2\tan^{-1}\sqrt{x-2}$.

5. $\dfrac{1}{b}\log\dfrac{\sqrt{x-a}-b}{\sqrt{x-a}+b}$.

6. $14\left(x^{\frac{1}{14}} - \frac{1}{2}x^{\frac{1}{7}} + \frac{1}{3}x^{\frac{3}{14}} - \frac{1}{4}x^{\frac{2}{7}} + \frac{1}{5}x^{\frac{5}{14}}\right)$.

Page 252. Art. 128

1. $2\log\left[1 - \sqrt{\dfrac{1+x}{1-x}}\right] + \dfrac{2(x-1)}{x-1+\sqrt{1-x^2}}$.

2. $-2\log\left[\sqrt{2} + \sqrt{\dfrac{2x-1}{x-1}}\right]$.

Pages 253–254. Exercises on Chapter IV

1. $2\sqrt{3}\tan^{-1}\sqrt{\dfrac{x^2-1}{3}} - \sqrt{x^2-1}$.

2. $\dfrac{3}{2}(x-a)^{\frac{1}{3}} - \dfrac{3}{4\sqrt{2}}\log\dfrac{\sqrt{2}(x-a)^{\frac{1}{3}}-1}{\sqrt{2}(x-a)^{\frac{1}{3}}+1}$.

3. $x - 4\sqrt{x+1} + 8\log(\sqrt{x+1}+2)$.

4. $\log(x+\sqrt{x-1}) - \dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2\sqrt{x-1}+1}{\sqrt{3}}$.

5. $\frac{3}{10}(2x-3a)(a+x)^{\frac{2}{3}}$.

6. $6\log(x^{\frac{1}{3}} - 3x^{\frac{1}{6}} + 5)$.

7. $\dfrac{1}{2\sqrt{2}a^2}\log\dfrac{\sqrt{x^2-a^2}+x\sqrt{2}}{\sqrt{x^2-a^2}-x\sqrt{2}}$.

8. $\frac{1}{2}[x^2 - x\sqrt{x^2-1} + \log(x+\sqrt{x^2-1})]$.

9. $\frac{6}{7}x^{\frac{7}{6}} - \frac{6}{5}x^{\frac{5}{6}} + \frac{3}{2}x^{\frac{2}{3}} + 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} - 6x^{\frac{1}{6}} + 3\log(x^{\frac{1}{3}}+1) + 6\tan^{-1}x^{\frac{1}{6}}$.

10. $-\frac{1}{3}(2a^2+x^2)\sqrt{a^2-x^2}$.

11. $\sqrt{a^2-x^2} + \dfrac{a}{2}\log\dfrac{\sqrt{a^2-x^2}-a}{\sqrt{a^2-x^2}+a}$.

12. $-\dfrac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\dfrac{x}{a}$.

13. $\dfrac{\sqrt{x^2-a^2}}{a^2x}$.

14. $\dfrac{x}{a^2\sqrt{x^2+a^2}}$.

15. $\dfrac{x(2x^2+3a^2)}{3a^4(x^2+a^2)^{\frac{3}{2}}}$.

16. $\dfrac{1}{2}\log\dfrac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x}-\dfrac{\sqrt{a^2+x^2}}{x}$.

17. $\dfrac{x}{a^2\sqrt{a^2-x^2}}$.

Page 256. Art. 131

1. $\frac{1}{3}\tan^3 x+\tan x$.
2. $-\frac{1}{3}\cot^3 x-\cot x$.
3. $\tan x+\frac{2}{3}\tan^3 x+\frac{1}{5}\tan^5 x$.
4. $-128\,[\cot 2x+\cot^3 2x$
   $+\frac{3}{5}\cot^5 2x+\frac{1}{7}\cot^7 2x]$.
5. $\frac{2}{3}\csc^3 x-\cot x-\frac{2}{3}\cot^3 x$.
6. $-64\,[\cot 4x+\frac{1}{3}\cot^3 4x]$.
7. $-\dfrac{1}{2\tan^2 x}+\log\tan x$.
8. $-\frac{1}{3}\cot^3 x-\frac{1}{5}\cot^5 x$.

Page 257. Art. 132

1. $\frac{1}{4}\sec^4 x-\frac{1}{2}\sec^2 x$.
2. $-\frac{1}{7}\csc^7 x+\frac{2}{5}\csc^5 x$
   $-\frac{1}{3}\csc^3 x$.
3. $\dfrac{1}{a}\left(\dfrac{1}{5}\sec^5 ax-\dfrac{2}{3}\sec^3 ax+\sec ax\right)$.
4. $-(\sin x+\csc x)$.
5. $\frac{1}{4}\sec^4 x-\sec^2 x+\log\sec x$.
6. $\dfrac{\sec^{n-1}x}{n-1}-\dfrac{\sec^{n-3}x}{n-3}$.
7. $\log\sec x$.
8. $-\log\csc x$.

Pages 259–260. Art. 133

1. $-\frac{1}{3}\cot^3 x+\cot x+x$.
2. $\dfrac{1}{2a}\tan^2 ax-\dfrac{1}{a}\log\sec ax$.
3. $\frac{1}{2}(\tan^2 x+\cot^2 x)$
   $+4\log(\sin x\cos x)$.
4. $\dfrac{\tan^{n-1}x}{n-1}$.
5. $\frac{1}{7}\tan^7 x-\frac{1}{5}\tan^5 x+\frac{1}{3}\tan^3 x$
   $-\tan x+x$.

Page 260. Art. 134 (*a*)

1. $-\cos x+\frac{1}{3}\cos^3 x$.
2. $-\frac{1}{5}\cos^5 x+\frac{1}{7}\cos^7 x$.
3. $\log\sin x-\sin^2 x+\frac{1}{4}\sin^4 x$.
4. $\frac{3}{4}\cos^{-\frac{4}{3}}x+3\cos^{\frac{2}{3}}x-\frac{3}{8}\cos^{\frac{8}{3}}x$.
5. $\frac{4}{3}(1-\cos x)^{\frac{3}{2}}-\frac{2}{5}(1-\cos x)^{\frac{5}{2}}$.

Page 261. Art. 134 (*b*)

1. $-\frac{1}{3}\cot^3 x$.
2. $-\cot x-\frac{2}{3}\cot^3 x-\frac{1}{5}\cot^5 x$.
3. $-\cot^5 x\,(\frac{1}{5}+\frac{1}{7}\cot^2 x)$.
4. $-\frac{1}{3}\cot^3 x-2\cot x+\tan x$.
5. $\frac{2}{3}\sqrt{\tan x}\,(\tan x-3\cot x)$.
6. $\dfrac{\tan^{n-1}x}{n-1}+\dfrac{\tan^{n+1}x}{n+1}$.

Page 262. Art. 134 (c)

1. $\frac{1}{8}x - \frac{1}{32}\sin 4x$.

2. $\frac{1}{128}(5x + \frac{8}{3}\sin^3 2x - \sin 4x - \frac{1}{8}\sin 8x)$.

3. $\frac{1}{128}(3x - \sin 4x + \frac{1}{8}\sin 8x)$.

4. $\frac{1}{8}(3x + \sin 4x + \frac{1}{8}\sin 8x)$.

5. $\tan x + \frac{1}{4}\sin 2x - \frac{3}{2}x$.

Page 264. Art. 134 (d)

1. $\frac{1}{2}(x - \sin x \cos x)$.

2. $\frac{1}{2}\cot x(\cos^2 x - 3) - \frac{3}{2}x$.

3. $-\frac{1}{3}\sin^3 x - \sin x + \log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$.

4. $-\frac{1}{4}\cot 2x \csc 2x + \frac{1}{4}\log\tan x$.

5. $\frac{5}{2}x + 2\cot x + \frac{1}{2}\sin x\cos x - \frac{1}{3}\cot^3 x$.

Page 265. Art. 135

1. $\frac{1}{4}\tan^{-1}(\frac{1}{2}\tan x)$.

2. $\frac{1}{2}\tan^{-1}\left[2\tan\left(\frac{x}{2} - \frac{\pi}{4}\right)\right]$.

3. $\frac{1}{2\sqrt{3}}\log\frac{\tan x - 2 - \sqrt{3}}{\tan x - 2 + \sqrt{3}}$.

4. $\frac{1}{\sqrt{a^2+b^2}}\log\frac{b\tan\frac{x}{2} - a + \sqrt{a^2+b^2}}{b\tan\frac{x}{2} - a - \sqrt{a^2+b^2}}$.

5. $\frac{-1}{a(a\tan x + b)}$.

6. $\frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right)$.

7. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)$.

8. $\frac{1}{2}\log\frac{\tan\frac{x}{2} + 1}{\tan\frac{x}{2} - 3}$.

Pages 266–267. Exercises on Chapter V

3. $2\sqrt{\tan x}$.

4. $\frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x$.

5. $-\csc x + \log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$.

6. $\frac{1}{2}\tan^2 x\sin x + \frac{3}{2}\left[\sin x - \log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right]$.

7. $\sqrt{\frac{1+x}{1-x}}$.

8. $e^{\frac{x}{2}}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)$.

9. $-\frac{1}{5}e^{-x}(\sin 2x+2\cos 2x)$.

10. $\frac{1}{8}e^{2x}(2-\sin 2x-\cos 2x)$.

11. $\frac{1}{4}e^{x}(\sin x+\cos x-\frac{3}{5}\sin 3x - \frac{1}{5}\cos 3x)$.

15. $\frac{\sin^{n+1}x}{n+1}-\frac{\sin^{n+3}x}{n+3}$.

16. $\frac{2}{3}\tan^{\frac{3}{2}}x-2\sqrt{\cot x}$.

17. $-32\cot 2x(1+\frac{2}{3}\cot^2 2x+\frac{1}{5}\cot^4 2x)$.

18. $\frac{1}{4}\tan\frac{1}{2}x(1+\frac{2}{3}\tan^2\frac{1}{2}x+\frac{1}{5}\tan^4\frac{1}{2}x)$.

19. $\log\tan 2x$.

20. $-8[\cot 2x+\frac{1}{3}\cot^3 2x]$.

21. $-\frac{1}{3a^2}\left(\frac{\sqrt{a^2-x^2}}{x}\right)^3$.

22. $\frac{(a\alpha+b\beta)x}{\alpha^2+\beta^2}+\frac{b\alpha-a\beta}{\alpha^2+\beta^2}\log(\alpha\sin x+\beta\cos x)$.

23. $\frac{1}{2}\left(\frac{a}{\alpha}+\frac{b}{\beta}\right)x+\frac{1}{2}\left(\frac{a}{\alpha}-\frac{b}{\beta}\right)\log(\alpha e^x+\beta e^{-x})$.

24. $x\cos(a-b)+\sin(a-b)\log\sin(x+b)$.

Pages 281–283. Art. 142

2. $\frac{4}{3}p^{\frac{1}{2}}a^{\frac{3}{2}}$. 3. 2. 4. $36\sqrt{3}$. 5. $40\sqrt{5}$.

6. $\frac{3}{4}$. 7. $\frac{3}{2}\sqrt[3]{2}$. 8. $\frac{1}{3}$. 9. $\frac{7}{6}$. 10. πr^2. 11. $\frac{37}{12}$.

12. $4a^2\tan^{-1}\frac{x_1}{2a}$; $4\pi a^2$. 13. πab. 14. $\frac{135}{2}-72\log 2$.

15. $\frac{a^{x_2}-a^{x_1}}{\log a}$. 16. $\frac{24-8\sqrt{2}}{5}$. 17. $\frac{1}{6}$. 18. $\frac{8}{15}(b-a)^{\frac{5}{2}}$.

20. $\frac{32}{105}$. 21. $\frac{1}{2}$; $\frac{e^\pi+1}{2(e^\pi-1)}$. 22. $a^2\log n$. 23. 1. 25. $\frac{a^2}{6}$.

26. $\frac{8}{15}$. 27. $4ab\tan^{-1}\frac{b}{a}$. 28. $2\sqrt{2}$. 29. $\frac{\log 4}{\pi}$. 30. $\frac{2}{\pi}$; $\frac{1-e^{-\frac{\pi}{2}}}{\pi}$.

31. $\frac{a^2}{2}(4-\pi)$.

Pages 287–289. Art. 143

Ex. 4. $\frac{a^2}{2}$. 1. 4. 2. ∞. 3. 3. 4. $4a^2$.

5. ∞. 6. $3\pi a^2$. 7. 2π.

Pages 290–291. Art. 144

1. πab.
2. $\frac{3}{8}\pi a^2$.
3. $\frac{3}{2}$.
4. $\frac{3\pi}{4}$.
5. $\frac{3\pi(a^2-b^2)^2}{8ab}$.

Page 292. Art. 145

2. a^2.
3. πr^2.
4. $\frac{3\pi r^2}{2}$.
5. 25π.
6. $\frac{c}{2}(\rho_1-\rho_2)$.
7. $\frac{\pi a^2}{2}$.
8. $\frac{a^2\pi^3}{24}$.
9. $\frac{k}{2}\log\left(\frac{\theta_2}{\theta_1}\right)$.
10. $\frac{a^2}{n}$.
11. $\frac{1}{4}$.
12. $\frac{2}{3}$.
13. $\frac{3\pi}{8}$.
14. $\frac{\pi^5}{5}$.

Pages 296–297. Art. 148

In the following answers the values are given for Simpson's formula only, unless the trapezoidal formula is called for in the problem.

1. 22; 21.5.
3. 0.500014; 0.500002; 0.5000014; 0.5000011.
4. 5.2523.
5. 37.8555; 36.5261.
6. 0.9996.
7. 8.0047.
8. 39.6465.
9. 0.7593.
10. 0.7468.
11. 0.4443.
12. 335.
13. 3.006.
14. 1.1873; 1.1830; 11931.
15. 0.5633.

Pages 300–302. Art. 149

6. $\frac{1}{2}\pi ab$.
7. $\frac{1}{3}\pi ab$.
8. $\frac{1}{3}Ah$.
9. $\frac{\pi abp}{2}$.
10. $4\frac{1}{2}$ cu. ft.
11. $\frac{8}{3}a^3$.
12. $\frac{4}{3}\pi a^3\cos^4\theta$.
13. $\frac{1}{2}\pi abc^2$.
14. $\frac{8}{3}r^3$.

Pages 304–306. Art. 150

1. $\frac{\pi r^3}{6\sqrt{2}}(10-3\pi)$.
2. $2\pi\left[\frac{1}{3}(2r^2+a^2)\sqrt{r^2-a^2}-ar^2\sin^{-1}\frac{\sqrt{r^2-a^2}}{a}\right]$.
3. 7π.
4. $-\pi\left[\frac{1}{3}x_1^3+ax_1^2+4a^2x_1+8a^3\log\left(\frac{2a-x_1}{2a}\right)\right]; \infty$.

5. $2\pi^2a^3$. 6. $\dfrac{4\pi a^2b}{3}$. 7. $\frac{4}{3}\pi r^3$. 8. πk^3; ∞.

9. $\dfrac{32\pi a^3}{105}$. 10. $\dfrac{\pi a^3}{15}$. 11. $4\pi^2a^3$.

12. $\pi\left[8a^3\log\dfrac{2a}{y_1} - 4a^2(2a - y_1)\right]$; ∞. 13. $5\pi^2a^3$. 14. $\dfrac{8\pi a^3}{3}$.

Pages 308–309. Art. 151

1. $p[\sqrt{2} + \log(1+\sqrt{2})]$. 2. $\dfrac{61a}{216}$. 3. $\log\sqrt{3}$.

4. $6a$. 5. $2\pi r$. 6. $\dfrac{a}{2}(e - e^{-1})$. 7. $\frac{17}{12}$.

8. $2 - \sqrt{2} + \log\dfrac{1+\sqrt{2}}{\sqrt{3}}$. 9. $\dfrac{4(a^3 - b^3)}{ab}$. 10. $a\log 3 - \frac{1}{2}a$.

Page 310. Art. 152

1. $(\rho_2 - \rho_1)\sqrt{a^2+1}$. 2. $2\pi a$. 3. $8a$.

4. $a\left[\tan\dfrac{\theta}{2}\sec\dfrac{\theta}{2} + \log\tan\left(\dfrac{\theta+\pi}{4}\right)\right]_{\theta_1}^{\theta_2}$.

5. $\dfrac{a}{2}\left[\theta\sqrt{\theta^2+1} + \log(\theta + \sqrt{\theta^2+1})\right]_{\theta_1}^{\theta_2}$.

6. $\dfrac{(\pi^2+4)^{\frac{3}{2}}}{3} - \dfrac{8}{3}$. 7. $4 + \dfrac{2}{\sqrt{3}}\log(\sqrt{3}+2)$.

8. $\dfrac{3\pi a}{2}$. 9. $2a\left[\sqrt{5} - 2 - \sqrt{3}\log\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{2}(2+\sqrt{3})}\right]$.

Page 311. Art. 153

1. $8a$. 2. $\dfrac{8am}{m-1}$. 3. $6a$. 4. $\frac{1}{2}a\theta_1^2$.

5. $\frac{1}{2}(x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}})^{\frac{3}{2}} - \dfrac{a}{2}$. 6. $\sqrt{2}(e^{t_1} - 1)$.

7. $\frac{1}{27}[(4 + 9t_1^2)^{\frac{3}{2}} - 8]$.

Pages 313–315. Art. 154

1. $\pi a^2\left(1 - \dfrac{1}{e}\right)$. 3. $\frac{8}{3}\pi p^2(\sqrt{8} - 1)$.

2. $\pi(\pi - 2)$. 4. $\dfrac{\pi p^2}{2}[3\sqrt{2} - \log(1+\sqrt{2})]$.

5. (a) $2\pi b\left(b + \dfrac{a^2}{\sqrt{a^2-b^2}}\cos^{-1}\dfrac{b}{a}\right)$.

(b) $2\pi a^2 + \dfrac{\pi ab^2}{\sqrt{a^2-b^2}}\log\left[\dfrac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right]$.

6. $\frac{32}{5}\pi a^2$.
7. (α) $\pi b\sqrt{a^2+b^2}$; (β) $\pi a\sqrt{a^2+b^2}$.
8. $4\pi a^2$.
9. $\frac{64}{3}\pi a^2$.
10. $4\pi^2 ak$; $2\pi^2 a^2 k$.
11. $2\pi a^2(3\sin t_1 - 3t_1\cos t_1 - t_1^2\sin t_1)$.

Pages 316–320. Art. 155

2. $\rho = e^{\frac{\theta}{a}+C}$.
4. $\rho = -a\cos\theta + C$.
5. $y^2 = ax^2 + b$.
6. $\rho^n = c\sin n\theta$. Straight line. Cardioid.
7. $y = e^{ax+c}$.
8. $\rho = ce^{\frac{\theta}{a}}$.
9. $\dfrac{\pi a^3}{3}$.
10. $2\pi a^2$.
11. $a\log\dfrac{a}{y_1}$.
13. $\sqrt{a^2+b^2}(t_2-t_1)$.
14. 22.7 lb.
15. 0.9627 lb.
16. 4.4312.
17. $h = 5.28$ in.
18. $h = 43.17$ lb.

Pages 323–324. Art. 156

3. $xy = ay^2 + by - \frac{1}{2}$.
4. $y = kx(\log x - 1) + ax + b$.
5. $\frac{4}{15}kt^{\frac{5}{2}}$.

6. $y = 0.0002x^3 + 0.0036x + 1.12$, slope $= 0.0006x^2 + 0.0036$; $x = 20$, $y = 2.792$, slope $= 0.2436$.

7. $1000y = -0.046x^3 + 0.75x^2 - 2.05x + 40$; $1000y = \frac{1}{30}x^3 - 0.0575x^2 - 9.7x + 117.91\frac{2}{3}$.

Page 328. Art. 158

1. $xy + C$.
2. $-\cos x\cos y + C$.
3. Impossible.
4. $\log\dfrac{x}{y} + C$.
5. $x^3 + y^3 - 3axy + C$.
6. $\tan^{-1}\dfrac{x}{y} + C$.
7. $\frac{2}{3}x^3 + x^2y + 5x + \frac{1}{3}y^3 - \frac{1}{2}y^2 + C$.
8. $\frac{1}{5}x^5 + xy^4 + \frac{1}{3}x^3 - xy^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3 + 2y + C$.

Page 330. Art. 160

1. $\dfrac{1}{2\sqrt{2}}$.
2. 4.
3. $\frac{1}{2}$.
4. 1.
5. $\frac{4}{3}a^3$.
6. $6b^3$.
7. $\dfrac{\pi}{2}$.
8. $\frac{1}{6}\log 2$.

Pages 332–334. Art. 161

2. 64.
3. $4\frac{1}{6}$.
4. $4\pi - 8$.
5. 1.
6. $\frac{8}{15}$.
7. $\frac{\pi a^2}{8}$.
8. $\frac{3\pi a^2}{2}$.
9. a^2.
10. $2\int_0^{\frac{\pi}{2}}\int_{a(1-\cos\theta)}^{a}\rho d\rho d\theta$; $2\int_0^{a}\int_{\cos^{-1}\left(1-\frac{\rho}{a}\right)}^{\pi}\rho d\theta d\rho$; $2\int_{\frac{\pi}{2}}^{\pi}\int_{a}^{a(1-\cos\theta)}\rho d\rho d\theta$.
11. 20.
12. $\frac{8\pi}{3} - 2\sqrt{3}$.

Pages 336–337. Art. 162

2. $\frac{2a^3m}{3}$. 3. $\frac{16a^3}{3}$. 4. π. 5. $\frac{4}{3}\pi abc$. 6. $\frac{\pi ab}{2}$. 7. $\frac{1}{36}$. 8. $2\pi\sqrt{6}$.

Page 340. Art. 163

1. 6225 lb. 2. 1120.5 lb. 3. 9337.5 tons. 4. 66.4 lb. 5. 15.645 tons.

Pages 343–345. Art. 164

1. $(\frac{3}{2}, \frac{3}{4}, \frac{1}{2})$.
2. $(\frac{2}{5}, \frac{2}{5}, \frac{1}{15})$.
3. $\bar{x} = \frac{3(k+a)^2}{4(k+2a)}$, $\bar{y} = \bar{z} = 0$.
4. $\left(0, 0, \frac{3c}{8}\right)$.
5. $\frac{3}{8}\left[\frac{(a+h)^4 - a^4}{(a+h)^3 - a^3}\right]$ from center of sphere.
6. $\frac{3}{8}\left[\frac{7(r+a)^4 - 6r^4}{7(r+a)^3 - 6r^3}\right]$ from spherical center.
8. $\frac{ab^2 + 2a'bb' + a'^2b'}{2(ab + a'b')}$ above the base.
9. $\frac{4a\sin\frac{\theta}{2}}{3\theta}$ from the vertex.
10. $\frac{2(r^2 - a^2)^{\frac{3}{2}}}{3\left(r^2\cos^{-1}\frac{a}{r} - a\sqrt{r^2 - a^2}\right)}$ from center of circle.
11. $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$.
12. $\left(\frac{2a}{3(\pi - 2)}, \frac{2b}{3(\pi - 2)}\right)$.
13. $\left(\frac{2a}{5}, \frac{2a}{5}\right)$.
15. $\left(\frac{3a}{8}, \frac{3b}{8}, \frac{3c}{8}\right)$.

16. $\left(\frac{8a}{15}, \frac{16b}{15\pi}, \frac{16c}{15\pi}\right)$.

17. $\left(\frac{3a}{8}, \frac{3a}{8}, \frac{3a}{8}\right)$; $\bar{x} = \bar{y} = \bar{z} = \frac{3}{8}\left[\frac{(a+h)^4 - a^4}{(a+h)^3 - a^3}\right]$.

18. $\bar{x} = \bar{y} = \bar{z} = \frac{2r}{5}$.

19. $\left(0, 0, \frac{h}{4}\right)$, the base of the cone being in xy-plane. **20.** $(0, 0, \frac{1}{4})$.

21. (*a*) $a\cos^2\theta$; (*b*) $a\left(1 + \frac{\cos^4\theta}{1+\cos^2\theta}\right)$ both measured from the vertex.

Pages 347–348. Art. 165

1. $\frac{abc}{3}(b^2 + c^2)$.

2. $\frac{1}{2}\pi a^4 h$.

3. $\frac{\pi a^2 h}{12}(3a^2 + h^2)$.

4. $\frac{\pi a r^4}{10}$.

5. $\frac{1}{4}\pi abh(a^2 + b^2)$.

6. $\frac{\pi abh}{12}(3a^2 + 4h^2)$.

7. $\frac{\pi abh}{12}(3b^2 + h^2)$.

8. $\frac{8\pi r^5}{15}$; $\frac{8\pi}{15}[(r+h)^5 - r^5]$.

9. $\frac{8\pi r^{n+5}}{3(n+5)}$.

10. $\frac{1}{6}a^5$. **11.** $\frac{1}{2}\pi a(R^4 - r^4)$. **12.** $\frac{1}{3}a^3 b$. **13.** $\frac{1}{4}bh^3$. **14.** $\frac{1}{4}\pi a^4$

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